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**RESEARCH ARTICLE**

# EXISTENCE OF AN OPTIMAL SHAPE FOR A THERMOELASTICITY PROBLEM AND SHAPE DERIVATIVE VIA THE LAGRANGE METHOD

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**ABSTRACT**

In this paper, we examine shape optimization problems for thermoelasticity. We first propose a model of the thermoelasticity problem, and then provide a mathematical analysis for the model under consideration. We also show result for the existence of optimal shapes in three ways, and we conclude by giving a shape derivation result using the Lagrange method.

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# 1. INTRODUCTION

In this paper we deal with existence, form derivation and topological derivation results for a linear thermoelasticity problem. We note in the literature see [26], [27], [28], [29], [30] that thermoelasticity problems have many applications in solar energy, industry and renewable energies etc. This leads us to focus on existence and derivation results, which will be followed by applications in the above-mentioned fields in our future work. Domain optimization is used today in many industrial environments, Airbus for the reduction of structures, the improvement of resistance to vibrations and many other areas of physics. In [10], the authors address a shape optimization problem for a thermoelasticity model with uncertainties in the Robin boundary condition. The problem was formulated as the minimization of the volume of the body under an inequality constraint on the expectation of. They derived analytical expressions of the shape functional to obtain the shape derivative via second order correlations. An efficient numerical method based on the low rank approximation was proposed. The solution of the optimization problem was implemented numerically via the level method. The isogeometric approach has been adopted in research areas where sophisticated geometric representations are demanding, such as shell analysis [14, 16], fluid-structure interaction [12, 15], robust mesh [17], and shape design optimization [6,7]. With respect with thermoelastic behavior, the thermomechanical contact of the mortar problem [5, 6] and material distribution of functionally graded structures [9, 10], were studied using the isogeometric approach. For more information see [20, 23]. The paper in organized as follows: In the first section we give the introduction. In the second part we give some preliminaries results related to weak solution existence results for the studied linear thermoelasticity problem. In section 3, we study some existence results of shape optimization with constraint partial differential equations coming from the model in the stationary case. The section 4 is devoted to shape optimization using Lagrange method and the optimal condition is geven. And in the section 5, we give the conclusion and some extensions.

# 2. PRELIMINARIES RESULTS

Let $\Omega$ be a bounded domain in $R^3$, $\omega$ a subset of $\Omega$ and $\omega_\epsilon(x_0) = x_0 + \epsilon\omega$ a small set of size thus that $\bar{\omega}_\epsilon \subset \Omega$ for a given $x_0 \in \Omega$ and $\omega \subset \Omega$. Defining the characteristic function

Let $\Omega$ be a bounded domain in $R^3$, $\omega$ a subset of $\Omega$ and $\omega_\epsilon(x_0) = x_0 + \epsilon\omega$ a small set of size thus that $\bar{\omega}_\epsilon \subset \Omega$ for a given $x_0 \in \Omega$ and $\omega \subset \Omega$. Defining the characteristic function $1_{\Omega_\epsilon(x_0)} = 1_\Omega - 1_{\omega_\epsilon(x_0)}$ the perforated domain is given, following [2, 3, 5] as follows $\Omega_\epsilon = \Omega \backslash \bar{\omega}_\epsilon$. We define

$$H_{div(\Omega,\mathbb{R}^3)} = \{u_\Omega \epsilon (L^2(\Omega))^3, div(u_\Omega) \epsilon L^2(\Omega); \partial_n u_\Omega = 0, u_\Omega \epsilon H_0^1(\Omega)\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.1)$$

Considering a shape function J defined by

$$J(\Omega) = \alpha \int_{\Omega} |u_{\Omega} - u_0|^2 \, dx + \beta \int_{\Omega} \left|\nabla_{u_{\Omega}}\right|^2 dx, \qquad \text{(2.2)}$$

where $u_{\Omega} \in H_{div(\Omega, R^3)}$ is solution to the variational problem

$$\mu \int_{\Omega} \nabla u_{\Omega} . \nabla \upsilon \, dx - (\lambda + \mu) \int_{\Omega} \nabla(\nabla . u_{\Omega}) \upsilon \, dx - 3k\alpha \int_{\Omega} \nabla\theta_{\Omega} \upsilon \, dx = \int_{\Omega} f \upsilon \, dx \qquad \text{(2.3)}$$

and

$$\int_{\Omega} \nabla\theta_{\Omega} \nabla\varphi \, dx = \int_{\Omega} g\varphi \, dx \qquad \text{(2.4)}$$

for all $v \in H_{div(\Omega, R^3)}$ and $\phi \in D(\Omega)$ for some given functions $f \in H^1(R^3)$, $g \in H^{-1}(\Omega)$. In the case where $div\ u_{\Omega} = 0$ in $\Omega$, (2.3) is reduced to the following variational problem

$$\mu \int_{\Omega} \nabla u_{\Omega} . \nabla \upsilon \, dx = \int_{\Omega} f \upsilon \, dx + 3k\alpha \int_{\Omega} \nabla\theta_{\Omega} \upsilon \, dx \qquad \text{(2.5)}$$

with $u_{\Omega_\epsilon}$ is solution to:

$$\begin{cases} -\mu\Delta u_{\Omega_\epsilon} - (\lambda+\mu) div \overrightarrow{u\Omega_\epsilon}) + 3ka\nabla\theta_{\Omega} = \overrightarrow{f} \ \ in\ \Omega_\epsilon \\ -\Delta\theta_{\Omega_\epsilon} = g \ in\ \Omega_\epsilon \\ \frac{\partial\theta_{\Omega_\epsilon}}{\partial n} = h \ \ on\ \partial\Omega_\epsilon \\ \frac{\partial\theta_{\Omega_\epsilon}}{\partial n} = 0 \ on\ \partial_{\omega_\epsilon} \\ \frac{\partial u_{\Omega_\epsilon}}{\partial n} = v_0 \ on\ \partial\Omega_\epsilon \\ \frac{\partial u_{\Omega_\epsilon}}{\partial n} = 0 \ on\ \partial_{\omega_\epsilon} \end{cases} \qquad \text{(2.7)}$$

Where $u_{\Omega_\epsilon}$ *is* solution to the variational problem

$$\mu \int_{\Omega_\epsilon} \nabla u_{\Omega_\epsilon} . \nabla v dx - (\lambda + \mu) \int_{\Omega_\epsilon} \nabla(\nabla \cdot u_{\Omega_\epsilon}) v dx - 3k\alpha \int_{\Omega_\epsilon} \nabla\theta_{\Omega_\epsilon} v = \int_{\Omega_\epsilon} f v dx$$

***and***

$$\int_{\Omega_\epsilon} \nabla\theta_{\Omega_\epsilon} \nabla\varphi dx = \int_{\Omega_\epsilon} g\varphi dx.$$

*Existence of solution to (2.7)*

We consider the following note-by-(2.8) equation

$$\begin{cases} -\Delta\theta_{\Omega} = g \text{ in } \Omega \\ \frac{\partial\theta_{\Omega}}{\partial n} = h \text{ on } \partial\Omega \\ \frac{\partial\theta_{\Omega}}{\partial n} = 0 \text{ on } \partial\omega. \end{cases}$$

This will therefore lead us to study two boundary value problems separately. We will therefore seek a solution $\theta_{\Omega}$ of problem (2.8) that we will inject into the following systems of equations to study the solution of the problem

$$\begin{cases} -\mu\Delta u_{\Omega} - (\lambda + \mu)\nabla(div\ \vec{u}) + 3ka\nabla\theta_{\Omega} = \overrightarrow{f} \ \ in\ \Omega \\ \frac{\partial u_{\Omega}}{\partial n} = v_0\ on\ \partial\Omega \\ \frac{\partial u_{\Omega}}{\partial n} = 0\ on\ \partial_{\omega} \end{cases} \qquad \text{(2.9)}$$

We seek $\theta$ belonging to a Hilbert space solution of (2.8). We give the definition of what we mean by classical solution of this problem.

Definition 2.1 Given a bounded open set $\Omega$ of class $C^2$ of $R^N$ of boundary $\partial\Omega$ and $h \in C^0(\partial\Omega.)$ We call a classical solution to: problem (2.8) all function $u$ of class $C^2(\Omega)^-$ verifying

It is therefore clear that there is no uniqueness in the solution to this Neumann problem. In fact, if $u$ is a solution to this problem for any connected component $\Omega_0$ of $\Omega$ and any $c \in R$ then $u + c\chi_{\Omega 0}$ is still a classical solution. We also see that there is no solution for any function $h \in C^0(\partial\Omega)$ and $g \in C^0(\Omega)$.

$$\begin{cases} -\Delta u = g \text{ in } \Omega \\ \frac{\partial u(z)}{\partial n} = h,\ \forall\ z \in \partial\Omega \end{cases}$$

If the equation admits a classical solution then for any connected component $\Omega_0$ of $\Omega$, we have

$$h d\sigma + \int_{\Omega_0} g \, dx = 0 \quad \text{..........................(2.10)}$$

We have the following theorem:

Theorem 2.1 *Let $\Omega$ be a bounded open set of* $R^N$ *and $\partial\Omega$ its boundary of class* $C^2$. $h \in C^2(\partial\Omega)$ *and* $g \in L^2(\Omega)$.
*Neumann's problem (2.8) admits a solution if and only if,*

$$\int_{\partial\Omega} h d\sigma + \int_{\Omega} g \, dx = 0. \quad \text{..........................(2.11)}$$

Proof. See [7, 8].

Theorem 2.2 *Let $\Omega$ be a bounded open set of* $R^N$ *of class* $C^1$. *Then the following partial differential equation*

$$\begin{cases} -\mu\Delta u_\Omega - (\lambda+\mu)\nabla\left(div\overrightarrow{u\Omega}\right) + 3ka\nabla\theta_\Omega = \vec{f} \;\; in \;\; \Omega \\ \frac{\partial_{u_\Omega}}{\partial_n} = v_0 \; on \; \partial\Omega \\ \frac{\partial_{u_\Omega}}{\partial_n} = 0 \; on \; \partial_\omega \end{cases} \quad \text{..............................(2.12)}$$

*has a solution. This solution is unique up to an additive constant.*

**Proof**. See [7, 8].

# 3. OPTIMAL SHAPE RESULTS

### 3.1. Existence of a solution by monotonicity of the functional

For the existence of optimal shape, we adapt the method proposed by Buttazo and Dal Maso. For further information, the reader can consult [6]. This method shows the existence of a minimum by considering the functional as monotone for inclusion and lower semi-continuous for the topology of $\gamma$-convergence. We denote by $A(D)$ the set of quasi-open sets contained in a bounded open set $D \subset R^N$. It is defined as follows:

$A(D) = \{\Omega \subset D \backslash \Omega$ is quasi-open$\}$.

studying the follozing problem $\min\{G(\Omega) : \Omega \in A(D),\ |\Omega| = K,$ K constant$\}$ .............................. (3.1)

with $G$ the functional defined by $$G(\Omega) = \int_D p\left(x, u_\Omega, \nabla u_\Omega\right) dx. \quad \text{..............................(3.2)}$$

Under the constraint of the problem

$$\begin{cases} -\mu\Delta u_\Omega - (\lambda+\mu)\nabla(div\overrightarrow{u\Omega}) + 3ka\nabla\theta_\Omega = \vec{f} \;\; in \;\; \Omega \\ -\Delta\theta_\Omega = g \; in \; \Omega \\ \frac{\partial\theta_\Omega}{\partial_n} = h \;\; on \; \partial\Omega \\ \frac{\partial\theta_{\Omega_\epsilon}}{\partial_n} = 0 \; on \; \partial_\omega \\ \frac{\partial_{u_\Omega}}{\partial_n} = v_0 \; on \; \partial\Omega \\ \frac{\partial_{u_\Omega}}{\partial_n} = 0 \; on \; \partial_\omega \end{cases} \quad \text{..............................(3.3)}$$

and the Borel function $p$ verifies the following hypothesis:

$p(x,y,z)$ is lower semi continuous on $(x,z)$ in R for all $(x,z)$ in $D$ is decreasing on R p.p $(x,z) \in D$. There exists $C > 0$, $a \in L^1(D)$ thus that:

$$C\left(z^2 - bs^2 - a(x)\right) \leq p(x, s, z), \;\; \forall x, s, z$$

with $b$ is a positive constant. Furthermore, we assume that the function $p$ is decreasing. With this hypothesis, we can give the existence of an optimal domain $\Omega$. This optimal domain belongs to the class of $p - quasi$ open sets, defined as the sets $\{u > 0\}$ for some function $u \in W_0^{1,p}(D)$. As a consequence, if $p > N$ these optimal sets are actually open, but if $p \leq N$ this fact does not occur any more under the very general assumptions we made. The existence of optimal sets $\Omega$ could have been obtained through a generalization the case $p > 1$, making use of a $\gamma_p$–convergence on the class of p-quasi open sets.

We can prove, under rather general assumptions on the integrand $p$, that $\Omega$ has a finite perimeter. For more informations the reader can refer to [21].

**Theorem 3.1** *Let $G$*: $A(D) \to (-\infty,+\infty)$ *be a functional defined by*

$$G(\Omega) = \int_D p(x, u_\Omega, \nabla u_\Omega)\, dx \quad \text{.................................(3.4)}$$

*with $u_\Omega$ solution of the problem (3.3) and p verifies the above hypothesis. Then the problem (3.1) admits a solution for all K* > 0.

**Proof.**

We start by showing the lower semi-continuity of the functional.

To do this, we set $m = \inf\{G(\Omega),\ \Omega \in A(D)\}$ and $u_{\Omega n} = u_n$. Since $u_n$ is a solution of (3.3), then $m > -\infty$, and thus there exists a minimizing sequence $(\Omega_{n_k})$ contained in A(*D*) such that $G(\Omega) \to m$.

Consider $u_n$ a solution of the following problem:

$$\begin{cases} -\mu\Delta u_n-(\lambda+\mu)\nabla(div\overrightarrow{u_n})+3k\alpha\nabla\theta_n=\vec{f} \ \ in\ \Omega_n \\ -\Delta\theta_\Omega=g\ in\ \Omega_n \\ \frac{\partial\theta_{\Omega n}}{\partial n}=h\ \ on\ \partial\Omega_n \\ \frac{\partial\theta_{\Omega n}}{\partial n}=0\ on\ \partial_{\omega n} \\ \frac{\partial u_{\Omega n}}{\partial n}=v_0\ on\ \partial\Omega_n \\ \frac{\partial u_{\Omega n}}{\partial n}=0\ on\ \partial_{\omega n} \end{cases} \quad \text{… … … … … … … … …. . … … … …. (3.5)}$$

We defined the function $\widetilde{u_n}$ by

$$\tilde{u}_n = \begin{cases} u_n & if\ x \in\ \Omega_n \\ 0 & if\ x \in D \setminus \Omega_n . \end{cases}$$

Now, from the variational formula, we get

$$\mu\int_D \nabla\widetilde{u_n}.\nabla v\, dx - (\lambda+\mu)\int_D \nabla(\nabla.\widetilde{u_n})v\, dx - 3k\alpha\int_D \nabla\theta_\Omega v = \int_D f v\, dx. \quad \text{……………………………(3.6)}$$

Taking $v = \widetilde{u_n}$ in (3.6), we get

$$\mu\int_D |\nabla\widetilde{u_n}|^2\, dx + \mu\int_D |\nabla.\widetilde{u_n}|^2\, dx \le ||f||^2\ \ ||\widetilde{u_n}||^2\ \ + 3k\alpha||\nabla\theta_{\Omega\, nk}||^2\ \ ||\widetilde{u_n}||^2 \quad \text{……………………(3.7)}$$

The solution $\theta_{\Omega_{nk}}$ of the Laplacian operator is also bounded in $H^1(D)/\mathbb{R}$, then the term $||\nabla\theta_{\Omega\, nk}||^2$ is also finite. So there exists a constant $\widetilde{M}$ depending on *f, k, α* and *μ* such that $||u_n||_{H_{div}} \le \widetilde{M}$.

The sequence $u_{\Omega_{nk}}$ is bounded in $H_{div(\Omega,R^3)}(D)$, which is a reflexive space. Then there exists an extracted sequence of $(u_{\Omega_{nk}})$ still noted by $(u_{\Omega\, nk})_{k\ge1}$ and $u_\Omega$ such that:

$$(u_{\Omega_{n_k}})_{k\geq1} \rightharpoonup u^*_\Omega \in L^2(\Omega_{n_k}),\ \ (\nabla u_{\Omega_{n_k}})_{k\geq1} \rightharpoonup \nabla u^*_\Omega \in L^2(\Omega), \quad \text{………………………….(3.8)}$$

$$(div\, u_{\Omega_{n_k}})_{k\geq1} \rightharpoonup div\, u^*_\Omega \in L^2(\Omega),\ \ if\ \ k \longrightarrow \infty. \quad \text{………………………………(3.9)}$$

In the same way, as $||\nabla\theta_{\Omega\, nk}||^2$ is also bounded in $H^1(\Omega)/\mathbb{R}$, there exists also a subsequence $(\theta_{\Omega\, nk})_{k\ge1}$ such that the following convergence holds

$$(\theta_{\Omega\, nk})_{k\ge1} \to \theta_\Omega \in L^2(\Omega). \quad \text{………………………….. (3.10)}$$

Passing to the limit as $k \to \infty$ and using weak convergence

$$\mu\int_D \nabla u^*_\Omega.\nabla\varphi dx - (\lambda+\mu)\int_D \nabla(\nabla\cdot u^*_\Omega)\varphi dx - 3k\alpha\int_D \nabla\theta_\Omega\varphi = \int_D f\varphi dx. \quad \text{………………………..(3.11)}$$

Using Green formula in the first term of (3.11), we get

$$\int_D -\mu\Delta u^*_\Omega dx - (\lambda+\mu)\int_D \nabla(\nabla\cdot u^*_\Omega)\varphi dx - 3k\alpha\int_D \nabla\theta_\Omega\varphi = \int_D f\varphi dx\ \ \forall\varphi \in H_{div}(\Omega)$$

Since we have:

$$\begin{cases} -\mu\Delta u^*_\Omega - (\lambda+\mu)\nabla\left(div\overrightarrow{u^*_\Omega}\right) + 3k\alpha\nabla\theta_\Omega = \vec{f}\ in\ D \\ \dfrac{\partial u^*_\Omega}{\partial_n} = v\ on\ \partial D. \end{cases}$$

And so, $u_\Omega = u^*_\Omega$ is a solution of (3.3). Using the assumption that *p* is lower semi-continuous,

we have:

$$p(x, u^*_\Omega, \nabla u^*_\Omega) \leq \liminf p(x, \tilde{u}_{\Omega_{n_k}}, \nabla \tilde{u}_{\Omega_{n_k}}).$$

So upon integrating, we obtain:

$$\int_D p(x, u^*_\Omega, \nabla u^*_\Omega)\, dx \leq \lim_{k \to \infty} \int_D p\left(x, \tilde{u}_{\Omega_{n_k}}, \nabla \tilde{u}_{\Omega_{n_k}}\right) dx.$$

So, $G(\Omega) \leq \lim_{k \to \infty} \inf G(\Omega_{n_k})$. Therefore, the functional $G$ is lower semi-continuous with respect to the topology of $\gamma$-convergence. Next, we will show that $G$ is decreasing with respect to inclusion. Let $\Omega_1$ and $\Omega_2$ be two subsets of $R^3$ such that $\Omega_1 \subset \Omega_2$.

$$\begin{cases} -\mu\Delta u_1 - (\lambda+\mu)\nabla(div\vec{u_1}) + 3ka\nabla\theta_{\Omega 1} = \vec{f} \ \ in\ \Omega_1 \\ -\Delta\theta_\Omega = g\ in\ \Omega_1 \\ \frac{\partial\theta_{\Omega 1}}{\partial n} = h\ \ on\ \partial\Omega_1 \\ \frac{\partial\theta_{\Omega 1}}{\partial n} = 0\ on\ \partial_{\omega 1} \\ \frac{\partial u_{\Omega 1}}{\partial n} = v_0\ on\ \partial\Omega_1 \\ \frac{\partial u_{\Omega 1}}{\partial n} = 0\ on\ \partial_{\omega 1}. \end{cases} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3.12)$$

*and*

$$\begin{cases} -\mu\Delta u_2 - (\lambda+\mu)\nabla(div\vec{u_2}) + 3ka\nabla\theta_{\Omega_2} = \vec{f} \ \ in\ \Omega_2 \\ -\Delta\theta_\Omega = g\ in\ \Omega_2 \\ \frac{\partial\theta_{\Omega 2}}{\partial_n} = h\ \ on\ \partial\Omega_2 \\ \frac{\partial\theta_{\Omega 2}}{\partial_n} = 0\ on\ \partial_{\omega 2} \\ \frac{\partial_{u\Omega 2}}{\partial_n} = v_0\ on\ \partial\Omega_2 \\ \frac{\partial_{u\Omega 2}}{\partial_n} = 0\ on\ \partial_{\omega 2} \end{cases} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3.13)$$

we consider $u_{\Omega 2} - u_{\Omega 1}$ and we get:

$$\begin{cases} -\mu\Delta(u_{\Omega_2} - u_{\Omega_1}) - (\lambda+\mu)\nabla div(u_{\Omega_2} - u_{\Omega_1}) + 3k\alpha\nabla(\theta_{\Omega_2} - \theta_{\Omega_1}) = \vec{f}(u_{\Omega_2}) - f(u_{\Omega_1})\ in\ \Omega_1 \\ -\Delta(\theta_{\Omega_2} - \theta_{\Omega_1}) = 0\ in\ \Omega_1 \\ \frac{\partial(\theta_{\Omega 2} - \theta_{\Omega 1})}{\partial_n} = 0\ in\ \partial\Omega_1 \\ \frac{\partial(\theta_{\Omega 2} - \theta_{\Omega 1})}{\partial_n} = -\frac{\partial\theta_{\Omega 1}}{\partial_n}\ on\ \partial_{\omega 1} \\ \frac{\partial(u_{\Omega 2} - u_{\Omega 1})}{\partial_n} = 0\ on\ \partial\Omega_1 \\ \frac{\partial(u_{\Omega 2} - u_{\Omega 1})}{\partial_n} = -\frac{\partial u_{\Omega 2}}{\partial_n}\ on\ \partial_{\omega_1}. \end{cases} \qquad (3.14)$$

By using the maximum principle, we have $u_{\Omega 2} \geq 0$ in $\Omega_2$, and therefore, by the same principle, we also have $(u_{\Omega 2} - u_{\Omega 1})(x) \geq 0$ for all $x \in \Omega_1$. We also have the assumption that $p(x,y,z)$ is decreasing, thus: $p(x,u_{\Omega 1},\nabla u_{\Omega 1}) \geq p(x,u_{\Omega 2},\nabla u_{\Omega 2})$. Thus we can conclude that $G(\Omega_1) \geq G(\Omega_2)$.

### 3.2. Existence of a solution by the -cone property

**Theorem 3.2** *Let* $\mathcal{O}_{ad} \subset \mathcal{O}_\epsilon$ *be a set open bounded domain of* $R^n$. *Then there exists on open set* $\Omega \in O_{ad}$ *satisfying*

$$J(\Omega) = \min_{\omega \in \mathcal{O}_{ad}} J(\omega).$$

**Proof.**

At first, we will find a lower bound for the functional $J(\Omega)$, $\Omega \in O_{ad}$. Because of the fact that $u_0 \in L^2(\Omega)$ and $u_\Omega$ is solution the problem (2.13), there exists a constant $M$ such that $0 \leq J(\Omega) \leq M$.

Then, the functional $J$ is bounded and there exists a minimizing sequence $\Omega_n \in O_{ad}$ such that $J(\Omega_n) \to m = \inf J(\Omega)$.

As $\Omega_n \in O_{ad}$ and $O_{ad}$ is closed to O, then according to a compactness theorem, there exists an open set $\Omega \in O_{ad}$ and a subsequence $\Omega_{n_k}$ of $\Omega_n$ such that the following convergence holds:

$$u_{\Omega_{n_k}} \xrightarrow{H} u_\Omega, \qquad \chi_{\Omega_{n_k}} \xrightarrow{L^1 p.p} \chi\Omega$$

$$\overline{\Omega_{n_k}} \xrightarrow{H} \overline{\Omega}, \qquad \partial\Omega_{n_k} \xrightarrow{H} \partial\Omega.$$

*Now, from the variational formula, we get*

$\mu\int_{\Omega}\nabla u_{\Omega}.\nabla v\, dx-(\lambda+\mu)\int_{\Omega}\nabla(\nabla.u_{\Omega})v\, dx-3k\alpha\int_{\Omega}\nabla\theta_{\Omega}v\, dx=\int_{\Omega}fv\, dx.$ ………………………………………….. (3.15)

Hence in $\Omega_{nk}$ we get:

$\mu\int_{\Omega_{nk}}\nabla u_{\Omega_{nk}}.\nabla v\, dx-(\lambda+\mu)\int_{\Omega_{nk}}\nabla(\nabla.u_{\Omega_{nk}})v\, dx-3k\alpha\int_{\Omega_{nk}}\nabla\theta_{\Omega_{nk}}v\, dx=\int_{\Omega_{nk}}fv\, dx$ ……………………………………..(3.16)

Taking $v=u_{\Omega_{nk}}$ in (3.16), we get

$\mu\int_{\Omega_{nk}}|\nabla u_{\Omega_{nk}}|^2\, dx+(\lambda+\mu)\int_{\Omega_{nk}}|\nabla.u_{\Omega_{nk}}|^2\, dx-3k\alpha\int_{\Omega_{nk}}\nabla\theta_{\Omega_{nk}}u_{\Omega_{nk}}=\int_{\Omega_{nk}}fu_{\Omega_{nk}}\, dx$ … … … … … … … … … … … … …(3.17)

$\mu\int_{D}\chi_{\Omega_{n_k}}|\nabla u_{\Omega_{nk}}|^2+\mu\int_{D}\chi_{\Omega_{n_k}}|\nabla.u_{\Omega_{nk}}|^2\, dx\leq\int_{D}\chi_{\Omega_{n_k}}fu_{\Omega_{nk}}\, dx+3k\alpha\int_{D}\chi_{\Omega_{n_k}}\nabla\theta_{\Omega_{nk}}u_{\Omega_{nk}}dx$

$\leq||f||^2\ ||u_{\Omega_{nk}}||^2\ +3k\alpha||\nabla\theta_{\Omega_{nk}}||^2\ ||\nabla u_{\Omega_{nk}}||^2$ . ……………………………(3.18)

From (3.17),

as $u_{\Omega_{nk}}$ is bounded in *D,* we have

The solution $\theta_{\Omega_{nk}}$ of the Laplacian operator is also bounded in $H^1(D)/R$, then the term $||\nabla\theta_{\Omega_{nk}}||^2$ is also finite, so there exists a constant $\tilde{M}$ depending on *f, k,*$\alpha$ and $\mu$ such that

$||u_{\Omega_{nk}}||_{H_{div}}\leq\tilde{M}.$

The sequence $u_{\Omega_{nk}}$ is bounded in $H_{div}$ *(D*), which is a reflexive space. Then there exists an extracted sequence of $(u_{\Omega_{n_k}})$ still denoted by $(\nabla u_{\Omega_{nk}})_{k\geq1}$ and $u_\Omega$ such that: if $k\rightarrow\infty$,

$u_{\Omega_{nk}}\rightarrow u^*\in L^2(\Omega_{n_k})$, $(\nabla u_{\Omega_{nk}})_{k\geq1}\rightarrow\nabla u^*\in L^2(\Omega)$, $(\text{div}\, u_{\Omega_{nk}})_{k\geq1}\rightarrow div\, u^*\in L^2(\Omega)$ ……………………………(3.19)

In the same way, as $||\nabla\theta_{\Omega_{nk}}||^2$ is also bounded in $H^1(\Omega)/R$, there exists also a subsequence $(\theta_{\Omega_{nk}})$ such that the following convergence holds $(\nabla\theta_{\Omega_{nk}})_{k\geq1}\rightarrow\nabla\theta_\Omega\in L^2(\Omega)$. ……………………………..(3.20)

From (3.16), we get

$$\mu\int_D\chi_{\Omega_{n_k}}\nabla u_{\Omega_{n_k}}.\nabla\varphi dx-(\lambda+\mu)\int_D\chi_{\Omega_{n_k}}\nabla(\nabla\cdot u_{\Omega_{n_k}})\varphi dx-3k\alpha\int_D\chi_{\Omega_{n_k}}\nabla\theta_{\Omega_{n_k}}\varphi=\int_D\chi_{\Omega_{n_k}}f\varphi dx.$$

Passing to the limit as $k\rightarrow\infty$ and using weak convergence (3.19) and (3.20), we get the following formulation

$$\mu\int_D\chi_\Omega\nabla u^*_\Omega.\nabla\varphi dx-(\lambda+\mu)\int_D\chi_\Omega\nabla(\nabla\cdot u^*_\Omega)\varphi dx-3k\alpha\int_D\chi_\Omega\nabla\theta_\Omega\varphi=\int_D\chi_\Omega f\varphi dx \qquad \text{…………………………(3.21)}$$

giving the following weak formulation

$$\mu\int_\Omega\nabla u^*_\Omega.\nabla\varphi dx-(\lambda+\mu)\int_\Omega\nabla(\nabla\cdot u^*_\Omega)\varphi dx-3k\alpha\int_\Omega\nabla\theta_\Omega\varphi=\int_\Omega\chi_\Omega f\varphi dx. \qquad \text{…………………………(3.22)}$$

Using Green formula in the first term of (3.22), we get

$$\int_\Omega-\mu\Delta u^*_\Omega dx-(\lambda+\mu)\int_\Omega\nabla(\nabla\cdot u^*_\Omega)\varphi dx-3k\alpha\int_\Omega\nabla\theta_\Omega\varphi=\int_\Omega f\varphi dx\quad\forall\varphi\in H_{div}(\Omega).$$

since we have:

$$\begin{cases}-\mu\Delta u^*_\Omega-(\lambda+\mu)\nabla(div\overrightarrow{u^*_\Omega})+3k\alpha\nabla\theta_\Omega=\overrightarrow{f}\ \text{in}\ \Omega\\ \frac{\partial u^*_\Omega}{\partial n}=v\ \text{on}\ \partial\Omega\,.\end{cases}$$

Finally by taking $\varphi=u_{\Omega_{nk}}$ in (3.21), and $\varphi=u_\Omega$ in (3.22) we have:

$lim\left(\mu\int_{\Omega_{nk}}|\nabla u_{\Omega_{nk}}|^2dx-(\lambda+\mu)\int_{\Omega_{nk}}\nabla(\nabla.u_{\Omega_{nk}})u_{\Omega_{nk}}dx-3k\alpha\int_{\Omega_{nk}}\nabla\theta_{\Omega_{nk}}u_{\Omega_{nk}}\right)$

$=lim\int_{\Omega_{nk}}fu_{\Omega_{nk}}dx=\int_\Omega\text{f}(x)u^*_\Omega$

$=\mu\int_\Omega|\nabla u^*_\Omega|^2dx-(\lambda+\mu)\int_\Omega\nabla(\nabla.u^*_\Omega)u^*_\Omega dx-3k\alpha\int_\Omega\nabla\theta_\Omega u^*_\Omega dx$

$\int_{\Omega_{nk}}|\nabla u_{\Omega_{nk}}-\nabla_{u_\Omega}|^2dx=\int_{\Omega_{nk}}|\nabla u_{\Omega_{nk}}|^2dx-2\int_{\Omega_{nk}}\nabla u_{\Omega_{nk}}\nabla_{u_\Omega}+\int_{\Omega_{nk}}|\nabla_{u_\Omega}|^2dx.$

Then taking the limit in the right hand side after equality, as $k \to \infty$

$$\int_{\Omega_{nk}} \left|\nabla u_{\Omega_{nk}} - \nabla_{u_\Omega}\right|^2 dx = \int_{\Omega_{nk}} \left|\nabla u_{\Omega_{nk}}\right|^2 dx - 2\int_{\Omega_{nk}} \nabla u_{\Omega_{nk}} \nabla_{u_\Omega} + \int_{\Omega_{nk}} \left|\nabla_{u_\Omega}\right|^2 dx = 0.$$

*Since*

$$\int_{\Omega_{nk}} \left|\nabla u_{\Omega_{nk}} - \nabla_{u_\Omega}\right|^2 dx = 0.$$

*And we show*

$$\int_{\Omega_{nk}} \left|\nabla(\nabla . u_{\Omega_{nk}}) - \nabla(\nabla . u_\Omega) u_\Omega\right|^2 dx = 0$$

$$\int_{\Omega_{nk}} f(u_{\Omega_{nk}} - u_\Omega) = 0 \quad and \quad \int_{\Omega_{nk}} u_{\Omega_{nk}} (\nabla\theta_{\Omega_{nk}} - \nabla\theta_\Omega) = 0.$$

*We have:*

$$u_{\Omega_{nk}}(x) \xrightarrow{L^2} u_\Omega(x)$$

$$\nabla(\nabla . u_{\Omega_{nk}}) u_{\Omega_{nk}} \xrightarrow{L^2} \nabla(\nabla . u_\Omega) u_\Omega$$

$$\nabla\theta_{\Omega_{nk}} \xrightarrow{L^2} \nabla\theta_\Omega.$$

*Since:*

$$J(\Omega_{nk}) \to J(\Omega).$$

We prove that $\Omega^*$ is a minimizer of $J$.

**Existence of a solution by compactness of set**

Here we will weaken the assumptions but nevertheless the functional $J$ remains lower $\gamma$−semi continuous for the topology of $\gamma$−convergence and we study the compactness of $A(D)$ for this convergence. The idea is to do a penalization of the functional $J$. This gives us $F(\Omega) = J(\Omega) + \alpha_1[|\Omega| - c]^+$ where $\alpha_1 \in R^+$ is a penalization factor and $J(\Omega) = \beta\int_\Omega |\Delta u_\Omega|^2 dx + \alpha\int_\Omega |u_\Omega - u_0|^2 dx$.

In this subsection, we will study a problem of the type

$$\min \{F(\Omega) : \Omega \in A, |\Omega| \leq c\}, \quad \text{.......(3.23)}$$

with $F : A(D) \to R^-$ a constrained shape functional of a parabolic boundary problem with $u_\Omega$ as its solution and is defined by

$$\begin{cases} -\mu\Delta u_\Omega - (\lambda+\mu)\nabla(div\overrightarrow{u_\Omega}) + 3ka\nabla\theta_\Omega = \vec{f} \quad in\ \Omega \\ -\Delta\theta_\Omega = g\ in\ \Omega \\ \frac{\partial\theta_\Omega}{\partial_n} = h\ \ on\ \partial\Omega \\ \frac{\partial\theta_\Omega}{\partial_n} = 0\ on\ \partial_\omega \\ \frac{\partial_{u_\Omega}}{\partial_n} = v_0\ on\ \partial\Omega \\ \frac{\partial_{u_\Omega}}{\partial_n} = 0\ on\ \partial_{\omega.} \end{cases} \quad \text{.......(3.24)}$$

**Theorem 3.3** *Let* $F$: $A(D) \to (-\infty,+\infty]$ *be a shape functional that is lower* $\gamma$ − *semi* − *continuous weak. Then the following problem*

$$\min\{F(\Omega) : \Omega \in A(D)\} \quad \text{.......(3.26)}$$

*Has a solution.*

**Proof.**

We start by showing the lower semi-continuity of the functional. To do this, we set $m = \inf\{G(\Omega), \Omega \in A(D)\}$ and $u_{\Omega n} = u_n$. Since $u_n$ is a solution of (3.25), then $m > -\infty$, and thus there exists a minimizing sequence $(\Omega_{n_k})$ contained in $A(D)$ such that $G(\Omega) \to m$.

Consider $u_n$ as a solution of the following problem:

$$\begin{cases} -\mu\Delta u_n - (\lambda+\mu)\nabla(div\overrightarrow{u_n}) + 3ka\nabla\theta_n = \vec{f} \quad in\ \Omega_n \\ -\Delta\theta_{\Omega_n} = g\ in\ \Omega_n \\ \frac{\partial\theta_{\Omega_n}}{\partial_n} = h\ \ on\ \partial\Omega_n \\ \frac{\partial\theta_{\Omega_n}}{\partial_n} = 0\ on\ \partial_{\omega_n} \\ \frac{\partial u_{\Omega_n}}{\partial_n} = v_0\ on\ \partial\Omega_n \\ \frac{\partial u_{\Omega_n}}{\partial_n} = 0\ on\ \partial_{\omega_n}. \end{cases} \quad \text{.......(3.26)}$$

We defined the function $\widetilde{u_n}$ by

$$\widetilde{u_n} = \begin{cases} u_n & if\ x \in \Omega_n \\ 0 & if\ x \in D \setminus \Omega_n . \end{cases}$$

Now, from the variational formula, we get

$$\mu \int_D \nabla \widetilde{u_n}.\nabla v\, dx - (\lambda+\mu)\int_D \nabla(\nabla.\widetilde{u_n}) v\, dx - 3k\alpha \int_D \nabla\theta_\Omega v = \int_D f v\, dx. \quad \text{(3.27)}$$

Taking $v = \widetilde{u_n}$ in (3.27), we get

$$\mu \int_D |\nabla \widetilde{u_n}|^2\, dx + \mu \int_D |\nabla.\widetilde{u_n}|^2\, dx \le ||f||^2\ ||\widetilde{u_n}||^2\ + 3k\alpha ||\nabla\theta_{\Omega\, nk}||^2\ ||\widetilde{u_n}||^2 \quad \text{(3.28)}$$

The solution $\theta_{\Omega_{nk}}$ of the Laplacian operator is also bounded in $H^1(D)/R$, then the term $||\nabla\theta_{\Omega\, nk}||^2$ is also finite. So there exists a constant $\widetilde{M}$ depending on $f$, $k$, $\alpha$ and $\mu$ such that

$$||u_{\Omega_{nk}}||_{H_{div}} \le \widetilde{M}.$$

The sequence $u_{\Omega_{nk}}$ is bounded in $H_{div}$ $(D)$, which is a reflexive space. Then there exists an extracted sequence of $(u_{\Omega_{nk}})$ still noted by $(u_{\Omega_{nk}})_{k\ge1}$ and $u_\Omega$ such that:

$$(u_{\Omega_{n_k}})_{k\ge1} \rightharpoonup u^*_\Omega \in L^2(\Omega_{n_k}),\ \ (\nabla u_{\Omega_{n_k}})_{k\ge1} \rightharpoonup \nabla u^*_\Omega \in L^2(\Omega), \quad \text{(3.29)}$$

$$(div\, u_{\Omega_{n_k}})_{k\ge1} \rightharpoonup div\, u^*_\Omega \in L^2(\Omega),\ \ if\ \ k \longrightarrow \infty. \quad \text{(3.30)}$$

In the same way, as $||\nabla\theta_{\Omega\, nk}||^2$ is also bounded in $H^1(\Omega)/R$, there exists also a subsequence $(\theta_{\Omega\, nk})$ such that the following convergence holds

$$(\nabla\theta_{\Omega\, nk})_{k\ge1} \rightarrow \nabla\theta_\Omega \in L^2(\Omega). \quad \text{(3.31)}$$

Passing to the limit as $k \longrightarrow \infty$ and using weak convergence

$$\mu \int_D \nabla u^*_\Omega.\nabla\varphi dx - (\lambda+\mu)\int_D \nabla(\nabla\cdot u^*_\Omega)\varphi dx - 3k\alpha \int_D \nabla\theta_\Omega\varphi = \int_D f\varphi dx. \quad \text{(3.32)}$$

Using Green formula in the first term of (3.32), we get

$$\int_D -\mu\Delta u^*_\Omega dx - (\lambda+\mu)\int_D \nabla(\nabla\cdot u^*_\Omega)\varphi dx - 3k\alpha\int_D \nabla\theta_\Omega\varphi = \int_D f\varphi dx \quad \forall\varphi \in H_{div}(\Omega)$$

Since we have:

$$\begin{cases} -\mu\Delta u^*_\Omega - (\lambda+\mu)\nabla\left(div\overrightarrow{u^*_\Omega}\right) + 3k\alpha\nabla\theta_\Omega = \vec{f}\ in\ D \\ \dfrac{\partial u^*_\Omega}{\partial_n} = v\ on\ \partial D. \end{cases}$$

Since $\widetilde{u_n}$ is bounded in $H_{div}$ $(D)$ there exist $\widetilde{M} > 0$ thus that

$$||\widetilde{u_n}||_{H_{div}} \le \widetilde{M}.$$

$$\alpha\int_\Omega |u^*_\Omega - u_0|^2 dx + \beta\int_\Omega |\nabla u^*_\Omega|^2 dx \le \lim_{k\longrightarrow\infty}\inf\left[\alpha\int_\Omega |u_{\Omega_{nk}} - u_0|^2 dx + \beta\int_\Omega |\nabla u_{\Omega_{nk}}|^2 dx\right]$$

On the other hand, the lower semi-continuity of the Lebesgue measure leads

$$\alpha\int_\Omega |u^*_\Omega - u_0|^2\, dx + \beta\int_\Omega |\nabla u^*_\Omega|^2\, dx + \alpha_1(m_L(\Omega) - c) \le \lim_{k\to\infty} inf\left[\alpha\int_\Omega |u_{\Omega_{nk}} - u_0|^2\, dx + \beta\int_\Omega |\nabla u_{\Omega_{nk}}|^2\, dx + \lim_{k\to\infty}\alpha_1(m_L(\Omega_{n\,k}) - c)\right]$$

Then we have:

$$F(\Omega) \le \lim_{k\to\infty}\inf F(\Omega_{n\,k}).$$

Using Green formula in the first term of (3.32), we get

$$\int_D -\mu\Delta u^*_\Omega dx - (\lambda+\mu)\int_D \nabla(\nabla\cdot u^*_\Omega)\varphi dx - 3k\alpha\int_D \nabla\theta_\Omega\varphi = \int_D f\varphi dx \quad \forall\varphi \in H_{div}(\Omega)$$

Since we have:

$$\begin{cases} -\mu\Delta u^*_\Omega - (\lambda+\mu)\nabla\left(div\overrightarrow{u^*_\Omega}\right) + 3k\alpha\nabla\theta_\Omega = \vec{f}\ in\ D \\ \dfrac{\partial u^*_\Omega}{\partial_n} = v\ on\ \partial D. \end{cases}$$

Since $\widetilde{u_n}$ is bounded in $H_{div}$ $(D)$ there exist $\widetilde{M} > 0$ thus that

$||\widetilde{u_n}||_{H_{div}} \leq \widetilde{M}$.

$$\alpha \int_{\Omega} |u_{\Omega}^* - u_0|^2 dx + \beta \int_{\Omega} |\nabla u_{\Omega}^*|^2 dx \leq \lim_{k \to \infty} \inf \left[ \alpha \int_{\Omega} |u_{\Omega_{nk}} - u_0|^2 dx + \beta \int_{\Omega} |\nabla u_{\Omega_{nk}}|^2 dx \right]$$

On the other hand, the lower semi-continuity of the Lebesgue measure leads

$$\alpha \int_{\Omega} |u_{\Omega}^* - u_0|^2 \, dx + \beta \int_{\Omega} |\nabla u_{\Omega}^*|^2 \, dx + \alpha_1 (m_L(\Omega) - c) \leq \lim_{k\to\infty} inf \left[ \alpha \int_{\Omega} |u_{\Omega_{nk}} - u_0|^2 \, dx + \beta \int_{\Omega} |\nabla u_{\Omega_{nk}}|^2 \, dx + \lim_{k\to\infty} \alpha_1 (m_L(\Omega_{n_k}) - c) \right]$$

Then we have:

$F(\Omega) \leq \lim_{k\to\infty} \inf F(\Omega_{n_k})$.

# 4. Shape derivative result via Lagrange method

The objective of this section to calculate the shape derive of the functional (2.8). Before going further we first prove the following results which as useful for the main result. The idea is to use the celebrated method of Hadamard for the shape functional that we considered. This method was introduced by Hadamard in [9] and many other authors [22]. In there papers, the notions of shape derivative is given.

### *4.1. Shape optimization*

Let $\Omega \subset R^N$ be a bounded open set of class $C^2$. For $t \geq 0$, let $\Omega_t = \varphi_t(\Omega)$, where for all $t$, $\varphi_t$ associated for $V$ is diffeomorphism of $R^2$. These properties holds:

$$\dot{\phi}_0 = V, \ | \, det(\nabla(\phi_t)) \, | = j(t,x), \ \frac{d\phi_t}{dt} = -V, \ | \, det(\nabla \phi_t^{-1} \, | = j(-t,x)$$

Let $\Omega_t = (Id + V)(\Omega)$ be a bounded open set of class $C^2$. For $t \geq 0$, very small, and $V \in C^1 \cap W^{1,\infty}(R^2)$. Let us consider also, the function $J$ in $\Omega_t$. We have the following definition:

**Definition 4.1** One function $J(\Omega)$ of the domain is said to be shape differentiable at $\Omega$ if the mapping $t \rightarrow J(\Omega_t)$ from R into R is Frechet differentiable at $t = 0$. The corresponding Frechet derivative (or differential) is denoted by $DJ(\Omega, V)$ and the following expansion holds:

$J(\Omega_t) = J(\Omega) + tDJ(\Omega, V) + o(t)$.

In the following, consider also then functional defined in $\Omega_t$, by

$$J(\Omega t) = \beta \int_{\Omega_1} |\nabla u_{\Omega_1}|^2 \, dx + \alpha \int_{\Omega_1} |u_{\Omega_1} - u_0|^2 \, dx \qquad \text{...............................(4.1)}$$

where $u_t$ be the solution to the following problem

$$\begin{cases} -\mu\Delta u_{\Omega_1} - (\lambda+\mu)\nabla(div\overrightarrow{\ \ _{\Omega_1}}) + 3ka\nabla\theta_{\Omega_1} = \vec{f} \ \ in \ \Omega_1 \\ -\Delta\theta_{\Omega_1} = g \ in \ \Omega_1 \\ \frac{\partial\theta_{\Omega_1}}{\partial_n} = h \ \ on \ \partial\Omega_1 \\ \frac{\partial_{u_{\Omega_1}}}{\partial_n} = v \ on \ \partial\Omega_1. \end{cases} \qquad \text{... ... ... ... ... ... ... ... ... ... ...(4.2)}$$

We look, in this section for the shape derivative of the functional $J(\Omega)$. The key point in the calculation of the shape derivative $DJ(\Omega, V)$ is in general, the definition of an appropriate derivation for the mapping $\Omega \rightarrow u_\Omega$. This mapping has a Lagrangian derivative $\dot{u}_\Omega$ and an Eulerian derivative $u'_\Omega$ linking with the Laplacian derivative by

$u'(\Omega, V) = \dot{u}(\Omega) - \nabla u.V$.

For the definition of the Laplacian and eulerian derivative, we refer to [1], [23]. The following result is devoted to the shape derivative of the functional.

**Theorem 3.1** *Lets* $\Omega$ *a class domain* $C^1(\mathbb{R}^N)$ *and* $V$ *a class vector field* $C^1$.

*Let* $F \epsilon C^1((0,\epsilon)C^0(\overline{\Omega_t})) \cap C^0((0,\in), C^1(\overline{\Omega_t}))$. *The function defined by*

$J_1(\in) = \int_{\Omega_t} F(\in, x)dx$ is

differentiable and its derivative is given by : $DJ_1(\Omega_t, V) = \int_{\Omega_\in} \frac{\partial}{\partial \in} F(\in, x) + divF(\in, x)V(x))dx \quad (Hadamard)$

$DJ_1(\Omega_t, V) = \int_{\Omega_\in} \frac{\partial}{\partial \in} F(\in, x) + \int_{\partial\Omega_\in} F(\in, \sigma)V.nd\sigma$.

**Proof.** See [1]

**Theorem 4.1** *Let Ω be a domain of class $C^1(R^N)$ and J the functional defined by (4.1), where $u_{\Omega t}$ is solution* (4.2). *Then $J(\Omega_t)$ is differentiable and the shape derivative is given by*

$$DJ(\Omega;V) = 2\beta\int_\Omega |\nabla u_\Omega|.\nabla u' dx + 2\alpha\int_\Omega |u_\Omega - u_0|u'_\Omega dx + \alpha\int_{\partial\Omega}|u_\Omega - u_0|^2 V.n d\sigma + \beta\int_{\partial\Omega}|\nabla u_\Omega|^2 V.n d\sigma$$

where $u'_\Omega$, the shape derivative verifies

$$\begin{cases} -\mu\Delta u'_\Omega - (\lambda+\mu)\nabla(\nabla.u'_\Omega) - 3k\alpha\nabla\theta'_{\Omega_1} = 0 \quad in\ \Omega \\ \Delta\theta' = 0\ in\ \Omega \\ u' = -\nabla u.V = -\frac{\partial u}{\partial n}V.n\ on\ \partial\Omega. \end{cases} \qquad \text{……………………(4.3)}$$

**Proof.** In $\Omega_t$, the function $J$ is written as follows:

$$J(\Omega_t) = \beta\int_{\Omega_t}|\nabla u_\Omega|^2 dx + \alpha\int_{\Omega_t}|u_\Omega - u_0|^2 dx.$$

The function $J$ is differentiable and using Hadamard Formula we get

$$DJ(\Omega, V) = 2\beta\int_\Omega |\nabla u_\Omega|\nabla u'_\Omega dx + 2\alpha\int_\Omega |u_\Omega - u_0|u'_\Omega dx + \alpha\int_{\partial\Omega}|u_\Omega - u_0|^2 V.n d\sigma + \beta\int_{\partial\Omega}|\nabla u_\Omega|^2 V.n d\sigma$$

for any V vector field with $u'_\Omega$ the shape derivative of ut. We recall that, the mapping $\Omega_t \to u_{\Omega_t}$ has an Eulerian derivative, and in what follows, we look for the equation verifies by $u'_\Omega$. We give first the variational formula in Ωt. Multiplying the first equation of (4.2) by v ∈ $H_{div}$ (Ω), and integrating, we get

$$\mu\int_{\Omega_t}\nabla u_{\Omega_t}\nabla v dx - (\lambda+\mu)\int_{\Omega_t}\nabla(\nabla.u_{\Omega_t})v dx - 3k\alpha\int_{\Omega_t}\nabla\theta_{\Omega_t}v dx = \int_{\Omega_t} f(x)v(x)dx. \qquad \text{……………………………(4.4)}$$

For t quite small, we can differentiate (4.4) with ($v = \varphi$) fixed. By applying the formula of (Hadamard) we have:

$$\mu\int_\Omega \nabla u'_\Omega\nabla\varphi dx - (\lambda+\mu)\int_\Omega \nabla(\nabla.u'_\Omega)\varphi dx + \mu\int_{\partial\Omega}\nabla u_\Omega\nabla\varphi V.n d\sigma - 3k\alpha\int_\Omega \nabla\theta'_\Omega\varphi dx$$
$$-(\lambda+\mu)\int_{\partial\Omega}\nabla(\nabla.u_\Omega)\varphi V.ng d\sigma - 3k\alpha\int_{\partial\Omega}\nabla\theta_\Omega\varphi V.n d\sigma = \int_{\partial\Omega} f\varphi V.n d\sigma.$$

If $\varphi$ is null on the boundary (on a neighborhood of the edge), we have:

$$\int_\Omega [\mu\nabla u'_\Omega\nabla\varphi - (\lambda+\mu)\nabla(\nabla.u'_\Omega)\varphi - 3k\alpha\nabla\theta'_\Omega\varphi]\, dx = 0 \qquad \text{……………………………..(4.5)}$$

And we have

$$\mu\int_\Omega \nabla u'_\Omega\nabla\varphi dx = -\mu\int_\Omega \Delta u'_\Omega\varphi dx + \mu\int_{\partial\Omega}\frac{\partial u'_\Omega}{\partial n} = -\mu\int_\Omega \Delta u'_\Omega\varphi dx.$$

So (4.5) becomes

$$\int_\Omega [-\mu\Delta u'_\Omega\varphi - (\lambda+\mu)\nabla(\nabla.u'_\Omega)\varphi - 3k\alpha\nabla\theta'_\Omega\varphi]\, dx = 0 .$$

So

$$\int_\Omega [-\mu\Delta u'_\Omega - (\lambda+\mu)\nabla(\nabla.u'_\Omega) - 3k\alpha\nabla\theta'_\Omega]\, \varphi dx = 0 .$$

And we get:

$$-\mu\Delta u'_\Omega - (\lambda+\mu)\nabla(\nabla.u'_\Omega) - 3k\alpha\nabla\theta'_\Omega = 0 \text{ In } \Omega$$

in the sense of distributions.

Let us recover the condition at the boundary, using the equality:

$$u'(\Omega, V) = \dot{u}(\Omega) - \nabla u.V$$

The function $u_t o\ (Id + tV)$ defined on the domain fix Ω disappears on the boundary of Ω for any t. We can therefore deduce:

$$\frac{d}{dt}(u_t)\circ(Id+tV)|_{t=0} = \dot{u}(\Omega, V) = 0 \ on\ \partial\Omega .$$

In other words, $u_t \circ (Id+tV) \in H_0^1(\Omega)$ for any t, therefore, according to the equality:

$u'(\Omega, V) = \dot{u}(\Omega) - \nabla u.V$

$u'$satisfied

$u' = -\nabla u.V = -\frac{\partial u}{\partial n} V.n \ \ on\ \partial\Omega.$

The last egality comes from the fact that the gradient of $u$ is normal on the boundary.

*4.2. Optimal conditions*

**Theorem 4.2** *Let Ω be the solution of the shape optimization problem* min{*J*(Ω, *ω* ∈ $O_{ad}$, |*w*| = *c, c constant*} *under the constraint* $u_\omega$*solution to (2.6). Then, there exists a Lagrange multiplier* $\lambda = \lambda(\Omega)$ *such that*

$$k\,(\partial_n u_\Omega)^2 + \lambda(\Omega) = 0, \qquad \text{………………………………… (4.6)}$$

*where k is a constant.*

**Proof**. Assume that Ω is a minimizer of $J$ under the contraint $|\Omega| = c$, the theorem of Lagrange multipliers then implies that there exists a constant $\lambda$ such that for any group of diffeomorphisms $(\varphi t)t\in$R,

$$\frac{d}{dt}\left(J_f(\phi_t) + \lambda|\phi_t|\right) = 0 \qquad \text{for } t = 0.$$

Assume that $(\varphi_t)_t \in_{\mathrm{R}}$ is the flow associated with $V \in \mathcal{C}_0^\infty(\mathbb{R}^2)$. Then it is proved in the book of Antoine Henrot and Michel Pierre [1] that

Then

$$\left(\frac{d|\phi_t(\Omega_t)|}{dt}\right)_{t=0} = \int_{\partial\Omega} V.nd\sigma.$$

$$\left(\frac{dJ_f(\Omega_t)}{dt}\right)_{t=0} = -\frac{1}{2}\int_{\partial\Omega} (\partial_n u_\Omega)^2\, V.nd\sigma$$

The theorem of Lagrange then implies that there exists a constant $\lambda$ such that:

$$\frac{d}{dt}\left(J(\Omega_t) + \lambda|\phi_t|\right)_{t=0}$$

Thus

$$-\frac{1}{2}\int_{\partial\Omega} (\partial_n u_\Omega)^2\, V.nd\sigma + \lambda \int_{\partial\Omega} V.nd\sigma = 0$$

or

$$\int_{\partial\Omega} \left[-\frac{1}{2}(\partial_n u_\Omega)^2\, V.n + \lambda V.n\right] d\sigma = 0$$

or again

$$\int_{\partial\Omega} \left[-\frac{1}{2}(\partial_n u_\Omega)^2 + \lambda\right] V.nd\sigma = 0.$$

This gives us

$$-\frac{1}{2}(\partial_n u_\Omega)^2 + \lambda = 0.$$

Since $V$ is arbitrary, we infer that $\partial_n u_\Omega$ is constant on Ω. Moreover, since $u_\Omega \geq 0$ on Ω, by maximum principle, $\partial_n u_\Omega$ is positive.

# 5. CONCLUSION AND EXTENSIONS

In this paper, we have studied a thermoelasticity problem using optimization methods. We have shown the existence of optimal shape using γ–convergence with compactness of set, the monotonicity of a functional and in the other hand we using the epsilon-cone property. Then we have established the shape derivative using Lagrange method. We then plan to study the regularity problems as well as the numerical methods of this problem.

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