Available Online at http://www.journalajst.com

ISSN: 0976-3376

ASIAN JOURNAL OF SCIENCE AND TECHNOLOGY

Asian Journal of Science and Technology
Vol. 14, Issue, 09, pp. 12663-12673, September, 2023

RESEARCH ARTICLE

CHARACTERISTICS OF THE CATEGORY NATURAL DEDUCTION FOR LANGUAGE THE PROPOSITIONAL LOGIC SYSTEM

***Adel Mohammed Al-Odhari**

Faculty of Education, Humanities and Applied Sciences (khawlan) and Department of Foundations of Sciences, Faculty of Engineering, Sana'a University, 13509, Sana'a, Yemen

ARTICLE INFO

Article History:
Received 16th June, 2023
Received in revised form
06th July, 2023
Accepted 20th August, 2023
Published online 30th September, 2023

Keywords:

Category of Natural Deduction, Valid Arguments, Valid Proof, Inconsistency set.

ABSTRACT

Aim & In this paper, we introduce the third system which is called the category of natural deduction (CND) for language propositional logic system($LPLS$), it's essential different in its form and ways form truth -table and truth-tree for ($LPLS$), even itspick out (or select) the same symbolic language like that used in the system of truth-table and truth- tree for ($LPLS$). Anyway, with regard to truth -table and truth-tree for ($LPLS$), there is a terminology which is called algorithm due to mathematician Muhammed ibn Musa al-khwarizmi (780-850), from the meaning of algorithm, that is there are some mechanical procedures, that we commit to it leading us to right judgment relative to arguments. In this paper we make a new presentation method for (CND) for($LPLS$), the principle of the system of natural deduction is based on Garhard Gentzen, who investigated and published his work in 1934 and 1935 about it. In this article, we represent natural deduction as a category of natural deduction for language propositional logic systems, to study the characteristics of arguments in (CND) and investigate valid and invalid arguments, types of formulas, and relations between formulas and inconsistency sets.

Citation: *Adel Mohammed Al-Odhari.* 2023. "Characteristics of the category natural deduction for language the propositional logic system", *Asian Journal of Science and Technology*, 14, (09), 12663-12673.

INTRODUCTION

The aim of this paper is to complete our continuous work about publishing a series of new readings for symbolic logic. In [1] we introduced the features of truth - table for language propositional logic system. In [4] we displayed the truth- tree for propositional logic system (TTPLS), the current paper is devoted to the category of the natural deduction method as this system is related to (LPLS). Moreover, any logical system has some structure about rules of logical inference and axioms allow for us to derive a new formula from some given formulas. To see the conceptions of natural deduction principle, we refer to logical philosopher's books such as [Bergmann, pp 146-225], [Kahane, pp.52-111], [Pospessel, pp.59-149], [Forbes,pp. 86-145].

2. Category of Natural Deduction (CND) for ($LPLS$)

In this section, we will present the category of natural deduction for ($\boldsymbol{LPLS}$) which consists of the following:

Firstly, *Deduction*: *A rule of inference* is a mapping that maps a set (possible empty) of wffs, $\varphi_1, \varphi_2, \dots, \varphi_n$ into a wffω. It written as follows: $\varphi_1, \varphi_2 \dots, \varphi_n$ / $\therefore\ \omega$.

Secondly, *Derivational Rules* (DR) which consists of wffs have special name and given by:

1. ***Rules of Premises and Assumptions:*** Suppose that the set of premises $\mathrm{P} = \{A_j\!:\ j = 1,2,3, \dots, n\}$, the premises of an argument for are ordered in list at the start of the proof in the order in which they are given, each of them labeled by premises on the column of reason and numbered its line in the column of independence. The set of $A = \{B_j\!: j = 1,2, \dots, n\}$ has procedure likes premises but labeled by assumption such as the following matrix- table:

Table 2.1. Rules of Premises or Assumption

Inference Rules	Line -#	wff	Reason	Dependency
Premise.	(j)	A	Premise	$j.$
Assumption.	(k)	B	Assumption	$k.$

2. ***Rule of And-Elimination**/And-Introduction:* Suppose that the wff$\boldsymbol{A}$ $\boldsymbol{and}$ $\boldsymbol{B}$ with logical connective and. it occurs (or appears) in line say j, then at any later line say $\boldsymbol{k}$ may be infer $\boldsymbol{A}$ or $\boldsymbol{B}$ depend on what we went to prove or infer. Moreover, the rule of and elimination in some books known as by simplification rule. This rule symbolizes by $\wedge -$ Elimination. Also,if awff$\boldsymbol{A}$ in an argument form take place at a line j in structure of proof and a wff$\boldsymbol{B}$ comes at a line $\boldsymbol{k}$, then we could infer new formula $\boldsymbol{A \wedge B}$ at a line $\boldsymbol{i}$ labelling by $\boldsymbol{j}$,$\boldsymbol{k}$,$\wedge \boldsymbol{-I}$. Also, this rule is called conjunction rule. The schemata of $\wedge \boldsymbol{-E}$ and $\wedge \boldsymbol{-I}$ are given by the following matrix-table.

Table 2.2. Rules for $\wedge -$E and ,$\wedge -$I.

Inference Rules	Line -#	wff	Reason	Dependency
$\wedge -$ Elimination ($\wedge -E$).	(j)	$A \wedge B$		$a_1, \dots, a_n$.
	⋮			
	(k)	A	$j, \wedge -E$	$a_1, \dots, a_n$.
Also	(j)	$A \wedge B$		$a_1, \dots, a_n$.
	⋮			
	(k)	B	$j, \wedge -E$	$a_1, \dots, a_n$.
$\wedge -$ Introduction ($\wedge -I$).	(j)	A		$a_1, \dots, a_n$.
	⋮			
	(k)	B		$b_1, \dots, b_m$.
	⋮			
	(i)	$A \wedge B$	j,k,$\wedge -I$	$a_1, \dots, a_n, b_1, \dots, b_m$.

3. ***Rules of Arrow -Elimination \ Rule of Arrow -Introduction:*** If a wff $\boldsymbol{A \to B}$ happen in structure of proof at line $\boldsymbol{j}$ and a wff of $\boldsymbol{A}$ appear at line $\boldsymbol{k}$, then we deduced a wff of$\boldsymbol{B}$. Logician is called its by modus pones and we denoted by $\longrightarrow \boldsymbol{-Eilmination}$. In addition, if a wff $\boldsymbol{A}$ at a line $\boldsymbol{j}$ as an assumption or premises and a wff of $\boldsymbol{B}$ which inferred at a line $\boldsymbol{k}$ in structure of proof, then we can have deduced at a line $\boldsymbol{A \to B}$ at a line $\boldsymbol{i}$ labeling by $\boldsymbol{k}$,$\to \boldsymbol{-I}$.If $\boldsymbol{A}$ is an assumption in this case discharged it by put as antecedent. Next matrix-table shown the rules$\longrightarrow \boldsymbol{-E}$ and $\to \boldsymbol{-I}$.

Table 2.3. Rules for$\longrightarrow \boldsymbol{-E}$ and $\to \boldsymbol{-I}$.

Inference Rules	Line -#	wff	Reason	Dependency
$\to-$ Elimination ($\to -E$).	(j)	$A \to B$		$a_1, \dots, a_n$.
	⋮			
	(k)	A antecedent		$b_1, \dots, b_m$.
	⋮			
	(i)	B consequent	j,k $\to -E$	$a_1, \dots, a_n, b_1, \dots, b_m$.
$\to-$ Introduction ($\to -I$).	(j)	A	Pre or Ass	j.
	⋮			
	(k)	B		$b_1, \dots, b_m$.
	⋮			
	(i)	$A \to B$	j,k ,$\to -I$	$\{b_1, \dots, b_m\} - \{j\}$.

4. ***Rules of*** $\boldsymbol{\leftrightarrow}$***-Elimination \ $\boldsymbol{\leftrightarrow}$-Introduction*** **(biconditional)***:* If thewff$\boldsymbol{A \leftrightarrow B}$appear in the structure of proof at a line $\boldsymbol{j}$,then at line $\boldsymbol{k}$ can be deduce the wff$\boldsymbol{A \longrightarrow B \wedge B \longrightarrow A}$. By similar argument, if the wff occurs at line $\boldsymbol{j}$ and also the wff appear at line $\boldsymbol{k}$, then derive the wff$\boldsymbol{A \leftrightarrow B}$ at line $\boldsymbol{i}$. Next matrix-table shown the rules $\leftrightarrow \boldsymbol{-E}$ and $\leftrightarrow \boldsymbol{-I}$.

Table 2.4. Rules $\leftrightarrow \boldsymbol{-E}$ and ($\leftrightarrow \boldsymbol{-I}$).

Inference Rules	Line -#	**wff**	Reason	Dependency
$\leftrightarrow -$ Elimination ($\leftrightarrow -E$).	(j)	$A \leftrightarrow B$		$a_1, \dots, a_n$
	⋮			
	(k)	$A \longrightarrow B \wedge B \longrightarrow$A	j, ,$\leftrightarrow -E$	$b_1, \dots, b_m$.
$\leftrightarrow -$ Introduction ($\leftrightarrow -I$).	(j)	$A \longrightarrow B$		$a_1, \dots, a_n$
	⋮			
	(k)	$B \longrightarrow$A		$b_1, \dots, b_m$
	⋮			
	(i)	$A \leftrightarrow B$	j, k ,$\leftrightarrow -E$	$a_1, \dots, a_n$,$b_1, \dots, b_m$

5. ***Rules of$\vee$-Elimination \ $\vee$-Introduction (Disjunction):*** If awff$\boldsymbol{A \vee B}$ occurs at structure of proof at line $\boldsymbol{j}$ and assumed wff$\boldsymbol{A}$ at a line $\boldsymbol{k}$ and deduce a wff of$\boldsymbol{C}$ at a line $\boldsymbol{i}$ and the same times assumed a wff of$\boldsymbol{B}$ at line $\boldsymbol{g}$ and in later steps derive the same a wff of $\boldsymbol{C}$, then derive $\boldsymbol{C}$ in line $\boldsymbol{r}$ and labeled as shown in table. Moreover, if a wff $\boldsymbol{A}$ appear in line $\boldsymbol{j}$, then derive a wffof$\boldsymbol{A \vee B}$at line $\boldsymbol{k}$. the next matrix-table describe ($\vee \boldsymbol{-E}$) and ($\vee \boldsymbol{-I}$).

Table 2.5. Rules for ($\vee \boldsymbol{-E}$)and ($\vee \boldsymbol{-I}$).

Inference Rules	Line -#	**wff**	Reason	Dependency
$\vee -$ Elimination ($\vee -E$).	(j)	$A \vee B$		$a_1, \dots, a_n$.
	⋮			
	(k)	A	Ass	k.
	⋮			
	(i)	C		$b_1, \dots, b_m$.

	⋮			
Also	(g)	B	Ass	g
	⋮			
	(h)	C		$c_1, \dots, c_r$. and discharge by ($\vee -E$
	⋮			
	(r)	C	$j, k, I, g, h, \vee -E$	$\{a_1, \dots, a_n, b_1, \dots, b_m, c_1, \dots, c_r.\} - \{k, g\}$.
$\vee -$ Introduction ($\vee -I$).	(j)	A		$a_1, \dots, a_n$.
	⋮			
	(k)	$A \vee B$	$j, \vee -I$	$a_1, \dots, a_n$.
Or	(j)	A		$a_1, \dots, a_n$.
	⋮			
	(k)	$B \vee A$	$j, \vee -I$	$a_1, \dots, a_n$.

6. ***Rules of Negation-Elimination \ Negation-Introduction:***

Table 2.6. Rules $\neg - E$ and $\neg - I$.

Inference Rules	Line -#	wff	Reason	Dependency
$\neg -$ Elimination ($\neg - E$).	(j)	$\neg A$	Ass or pre	$a_1, \dots, a_n$
	⋮			
	(k)	A		$b_1, \dots, b_m$.
	⋮			
	(i)	0	j,k ,$\neg - E$	$a_1, \dots, a_n, b_1, \dots, b_m$.
$\neg -$ Introduction ($\neg - I$).	(j)	A	Ass or pre	$a_1, \dots, a_n$
	⋮			
	(k)	0		
	⋮			
	(i)	$\neg A$	j, k ,$\neg - I$	$a_1, \dots, a_n$-disch-j

7. Rules of Double-negation (DN) \ Double-negation $(DN)^+$-Introduction

Table 2.7. Rules for DN and $(DN)^+$.

Inference Rules	Line -#	**wff**	Reason	Dependency
DN	(j)	$\neg\neg A$		$a_1, \dots, a_n$
	⋮			
	(k)	A	j, DN	$a_1, \dots, a_n$.
$(DN)^+$.	(j)	A		$a_1, \dots, a_n$
	⋮			
	(k)	$\neg\neg A$	j,$(DN)^+$.	$a_1, \dots, a_n$

8. Rule of Ex-FalseQuodlibetic (EFQ) (or Absurdity)

Table 2.7. Rule for EFQ.

Inference Rules	Line -#	**wff**	Reason	Dependency
EFQ	(j)	O		$a_1, \dots, a_n$
	⋮			
	(k)	A	j,EFQ	$a_1, \dots, a_n$.

We expand the category natural deduction be adding theorems 5.2. and 5.4. which are deduced by second part in (CND).
Thirdly, Theorem 5.2. [Logical Implies] (TLI).
Fourthly, Theorem 5.4. [Provably Equivalent] (TPE).
Remark. Any assumption used in a proof must be discharged by the following method:

- An Assumption A is made an antecedent of a conditional **A ⟶ B**.
- The assumption A leads to a contradiction 0, then ¬A is consider.
- Assumption used as in the rule $\vee -$**E**.
- The contradiction and validity wff denoted by 0 and 1 respectively.

Observation about category natural deduction: There are two types of category natural deduction, namely,essential category natural deduction and nonessential category natural deduction according to the following definitions.

Definition 2.1. A category natural deduction rule is called essential if there is a valid proof such that category natural deduction rule cannot be proved its validity without using this proof. According to the pervious definition all Derivational Rules are essentials.

Definition 2.2. A category natural deduction rule is called nonessential (or immaterial) if there are no valid proofsthat require proof of its validity without use of that rule, more evidently, we can prove it by the category of natural deduction.

3.Classification Arguments (or proofs) in Natural Deduction for ($LPLS$)***:*** In this section, we introduce a third method to distinguish between valid \ invalid argument(or proof) upon the Category of natural deduction method for ($LPLS$). Recall that, in the truth-table of language of the propositional logic system, the argument is called valid if there is no valuation (or assignment) in the vertical line to determine the truth-value of "T" for its premises and the truth-value of "F" for its conclusion. One the other hand, in the truth- tree of language of the propositional logic systems, the argument is called valid if there is a set consisting of the premises and negate of conclusion such that this set is complete a closed tree. The next definition illustrates valid argument and meaning of proof in a new manner.

Definition 3.1. An *argument form* is a finite sequence of wffs $A_1, A_2, A_3, \dots, A_n$ is called *premises* followed by a wff$B$ called *conclusion.* This is written as follows:

$$A_1, A_2, A_3, \dots, A_n, \therefore B$$

Definition 3.2. Let $\boldsymbol{A_1, A_2, A_3, \dots, A_n, \therefore B}$ be an argument form, then it's called a *valid*, if there is a proof start by premises$\boldsymbol{A_1, A_2, A_3, \dots, A_n}$ and arrive to yields consequent $\boldsymbol{B}$. otherwise is called invalid.

Definition 3.3. A proof is a category consist of data: … with corresponding notation

1. **Objects: $\mathbf{P}, \mathbf{C}, \mathbf{D}$,**
2. **Rules of deduction: $\boldsymbol{r_1, r_2, \dots, r_n}$,** belonged to category of rules- deduction,
3. **Lines numbers: $\boldsymbol{l_1, l_2, \dots, l_m}$,** described the length of a proof,
4. **Reasoning procedures:** $p_1, p_2, \dots, p_m$, described 1 and 2 and
5. **Dependency**: $d_1, d_2, \dots, d_m$, described justification when transform between lines,where the objects $\mathbf{P} = \{A_i: , i = 1,2, \dots, n\}$ the set of **wffs** (premises), $\boldsymbol{C} = \{B: B, where\ B\ is\ \mathbf{wff}\}$ set of a conclusion and$\mathbf{D} = \{A_1, \dots, A_n, p_1, \dots, p_m, B\}$ set of sequent start from premises and the differential formulas by derivation-rulesand satisfying the following conditions:
   a. Every line contains **wff**belong to**D** .
   b. Conclusion formula member of $\boldsymbol{C}$,deduced lastly from all steeps of **D**,where $\mathbf{P}, \boldsymbol{C} \subset \boldsymbol{D}$**.**
   c. If step (b) holds, we say that a proof is valid, and denoted by:
   $\{A_1, \dots, A_n, l_1, \dots, l_m, p_1, \dots, p_m, d_1, \dots, d_m\} \vdash B$.(this is called sequent) and B is called a Theorem in ($LPLS$) if $\vdash B$. the structure of D represent the proof. Note that the set $\{A_1, \dots, A_n, l_1, \dots, l_m, p_1, \dots, p_m, d_1, \dots, d_m\}$ may be is an empty set, in this case $B$is a tautology wff as we will see later. We will use the symbol turnstile " $\vdash$" for proof, the definition of the word turnstile we quote from google translator is " a mechanical gate consisting of revolving horizontal arms fixed to a vertical post, allowing only one person at a time to pass through".

To classify the arguments let us assume that the order of the premises set P contains one wff (or an element). The next theorem illustrates how to deduce a wff, say $A$from different arguments.

Theorem 3.4.Consider thesequent of sets of wffs as the premises:

$\mathbf{P_1} = \{\neg\neg\boldsymbol{A}\}, \mathbf{P_2} = \{\boldsymbol{A} \wedge \boldsymbol{A}\}, \mathbf{P_3} = \{\ \boldsymbol{A} \wedge \mathbf{1}\}, \mathbf{P_4} = \{\ \boldsymbol{A} \wedge (\boldsymbol{A} \vee \boldsymbol{B})\}, \mathbf{P_5} = \{(\boldsymbol{A} \wedge \boldsymbol{B}) \vee (\boldsymbol{A} \wedge \boldsymbol{C})\}$,
$\mathbf{P_6} = \{\boldsymbol{A} \vee \boldsymbol{A}\}, \mathbf{P_7} = \{\boldsymbol{A} \vee \mathbf{0}\}, \mathbf{P_8} = \{\boldsymbol{A} \vee (\boldsymbol{A} \wedge \boldsymbol{B})\}, \mathbf{P_9} = \{\neg\boldsymbol{A} \rightarrow \boldsymbol{A}\}, \mathbf{P_{10}} = \{\boldsymbol{B}, \neg\boldsymbol{B}\}, \mathbf{P_{11}} = \{\mathbf{0}\}$
$\mathrm{P_{12}} = \{A \vee B, \neg B\}, \mathrm{P_{13}} = \{B, B \leftrightarrow A\}$ and set of a conclusion$\mathrm{C} = \{A\}$, then:
$\mathbf{P_1} \vdash \mathbf{C}, \mathbf{P_2} \vdash \mathbf{C}, \mathbf{P_3} \vdash \mathbf{C}, \dots, \mathbf{P_{13}} \vdash \mathbf{C}$.

Proof. With regarding to $\mathrm{P_1}, \mathrm{P_2}, \mathrm{P_3}, \mathrm{P_4}$, are justify by (CND-2) from $DN, \wedge -E$, Hence $\mathrm{P_1} \vdash \mathrm{C}, \mathrm{P_2} \vdash \mathrm{C}, \mathrm{P_3} \vdash \mathrm{C}$ and $\mathrm{P_4} \vdash \mathrm{C}$, the length of each proof is equal to 2. Also $\mathrm{P_6}$, $\mathrm{P_7}$,$\mathrm{P_8}$ and $\mathrm{P_9}$ by theorem (TPE) part 5,7,12 and 19. Therefore $\mathrm{P_6} \vdash \mathrm{C}, \mathrm{P_7} \vdash \mathrm{C}, \mathrm{P_8} \vdash \mathrm{C}$ and $\mathrm{P_9} \vdash \mathrm{C}$. With respect to $\mathrm{P_5}$ depend on ,$\vee -E$,while $\mathrm{P_{10}}$deducing by ,$\neg - E$,$\neg - I$ and DN. So $\mathrm{P_5} \vdash \mathrm{C}$ and $\mathrm{P_{10}} \vdash \mathrm{C}$ are both have length of proof is equal to 6. $\mathrm{P_{11}}$ infer by ,$\neg - I$ and DN with length 4. So $\mathrm{P_{11}} \vdash \mathrm{C}$. obviously,
$\mathrm{P_{13}}$ derive by ,$\leftrightarrow -E$,$\wedge -E$ and ,$\rightarrow -E$ with length of proof is equal to 4, thus $\mathrm{P_{13}} \vdash \mathrm{C}$. Finally, to illustrate the proof $\mathrm{P_{12}}$ as following:

Line #	wff	Reason	Dependency
1.	$A \vee B$	pre	1.wff-pre.
2.	$\neg B$	pre	2.wff-pre.
3.	$\neg A$	Ass	3.wff.ass.
4.	A	Ass	4.wff. ass.
5.	0	3,4,$\neg - E$	5.wff.der.
6.	B	Ass	6.wff.ass.
7.	0	2,6,$\neg - E$	7.wff.der.
8.	0	1,4,5,,6,7 $\vee -E$	Dis by $\vee -E$
9.	$\neg\neg A$	3,8,$\neg - I$	Dis by $\neg - I$
10.	A	9,DN	10.wff.der.

We note the length of the proof is equal to 10 and $\mathbf{P_{12}} \vdash \mathbf{C}$is to complete the proof ■.

Remark. Observation thata wff$\boldsymbol{A}$ is subformula from all sets of premises or a wff$\boldsymbol{A}$ deduced that from any other wff, say$\boldsymbol{B}$ and its negation. The next second theorem illustrates how to deduce a negation wff, say $\neg\boldsymbol{A}$ from different arguments.

Theorem 3.5. Consider the sequent of sets of wffs as the premises:

$\mathbf{P_1} = \{\boldsymbol{A} \longrightarrow \boldsymbol{B}, \neg\boldsymbol{B}\}, \mathbf{P_2} = \{\boldsymbol{A} \longrightarrow \boldsymbol{B}, \boldsymbol{A} \longrightarrow \neg\boldsymbol{B}\}, \mathbf{P_3} = \{\boldsymbol{A} \longrightarrow \boldsymbol{B}, \boldsymbol{B} \longrightarrow \neg\boldsymbol{A}\}, \mathbf{P_4} = \{\boldsymbol{A} \longrightarrow (\boldsymbol{B} \wedge \neg\boldsymbol{B})\}$,
$\mathrm{P_5} = \{A \longrightarrow B, (B \vee C) \longrightarrow D, D \longrightarrow \neg A\}, \mathrm{P_6} = \{(A \longrightarrow 0) \vee (B \longrightarrow 0), B\}, \mathrm{P_7} = \{\neg B, B \leftrightarrow A\}$ and set of a conclusion$\mathrm{C} = \{\neg A\}$, then: $\mathrm{P_1} \vdash \mathrm{C}, \mathrm{P_2} \vdash \mathrm{C}, \mathrm{P_3} \vdash \mathrm{C}, \dots, \mathrm{P_7} \vdash \mathrm{C}$.

Proof

$P_1 \vdash C$.			
Line #	wff	Reason	Dependency
1.	$A \longrightarrow B$	pre	1.wff-pre.
2.	$\neg B$	pre	2.wff-pre.
3.	A	Ass	3.wff.ass.
4.	B	$1,3, \longrightarrow -E$	4.wff.der.
5.	0	$2,4, \neg - E$	5.wff.der.
6.	$\neg A$	$3,5, \neg - I$	Dis-ass.
$P_2 \vdash C$.			
1.	$A \longrightarrow B$	pre	1.wff-pre.
2.	$A \longrightarrow \neg B$	pre	2.wff-pre.
3.	A	Ass	3.wff.ass.
4.	B	$1,3, \longrightarrow -E$	4.wff. der.
5.	$\neg B$	$2,3, \longrightarrow -E$	5.wff. der.
6.	0	$4,5, \neg - E$	6.wff. der.
7.	$\neg A$	$3,6, \neg - I$	Dis-ass.
$P_3 \vdash C$.			
1.	$A \longrightarrow B$	pre	1.wff-pre.
2.	$B \longrightarrow \neg A$	pre	2.wff-pre.
3.	A	Ass	3.wff.ass.
4.	B	$1,3, \longrightarrow -E$	4.wff. der.
5.	$\neg A$	$2,4, \longrightarrow -E.$	5.wff. der.
6.	0	$3,5, \neg - E.$	6.wff. der.
7.	$\neg A$	$3,6, \neg - I.$	Dis-ass.
$P_4 \vdash C$.			
1.	$A \longrightarrow (B \wedge \neg B)$	pre	1.wff-pre.
2.	A	Ass	2.wff.ass.
3.	$B \wedge \neg B$	$1,2, \longrightarrow -E.$	3.wff. der.
4.	B	$3 ,\wedge -E.$	4.wff. der.
5.	$\neg B$	$3,\wedge -E.$	5.wff. der.
6.	0	$4,5, \neg - E.$	6.wff. der.
7.	$\neg A$	$2,6, \neg - I.$	Dis-ass.
$P_5 \vdash C$.			
1.	$A \longrightarrow (B \wedge \neg B)$	pre	1.wff-pre.
2.	$(B \vee C) \longrightarrow D$	pre	2.wff-pre.
3.	$D \longrightarrow \neg A$	pre	3.wff-pre.
4.	A	Ass	4.wff.ass.
5.	$(B \wedge \neg B)$	$1,4, \longrightarrow -E.$	5.wff. der.
6.	B	$5,\wedge -E.$	6.wff. der.
7.	$B \vee C$	$6,\vee -I.$	7.wff. der.
8.	D	$2,7, \longrightarrow -E.$	8.wff. der.
9.	$\neg A$	$3,8, \longrightarrow -E.$	9.wff. der.
10.	0	$4,9, \neg - E.$	10.wff. der.
11.	$\neg A$	$4,10, \neg - I.$	Dis-ass.
$P_6 \vdash C$.			
1.	$(A \longrightarrow 0) \vee (B \longrightarrow 0)$	pre	1.wff-pre.
2.	B	pre	2.wff-pre.
3.	A	Ass	3.wff.ass.
4.	$A \longrightarrow 0$	Ass	4.wff.ass.
5.	0	$3,4, \longrightarrow -E.$	5.wff.der.
6.	$B \longrightarrow 0$	Ass	6.wff.ass.
7.	0	$2,6, \longrightarrow -E.$	7.wff.der.
8.	0	$1,4,5,6,7,\vee -E.$	$\{1,4,5,6,7\} - \{1\}$
9.	$\neg A$	$3,8,\vee -I.$	Dis-ass.3.
$P_7 \vdash C$			
1.	$\neg B$	pre	1.wff-pre.
2.	$B \longleftrightarrow A$	pre	2.wff-pre.
3.	A	Ass	3.wff.ass.
4.	$(B \longrightarrow A) \wedge (A \longrightarrow B)$	$2, \longleftrightarrow -E.$	4.wff.der.
5.	$B \longrightarrow A$	$3 ,\wedge -E.$	5.wff.der.
6.	$A \longrightarrow B$	$3 ,\wedge -E.$	6.wff.der.
7.	B	$3,6 , \longrightarrow -E.$	7.wff.der.
8.	0	$1,7 , \neg - E.$	8.wff.der.
9.	$\neg A$	$3,8 , \neg - I.$	Dis-ass.3.

The proof is complete■.

4.Determines Types of wffs in (CNDM) for ($LPLS$): The (CNDM) for ($LPLS$)has no ability to decide on any concept that is defined by using the idea of probability, since the contingence wff defined on concept of its, so that (CNDM) for ($LPLS$) is ignoring it, in contracts, the wff of tautology and contradiction not related to the probability concepts is allows to (CNDM) for ($LPLS$) to decide about them.

Definition 4.1.A wff $\boldsymbol{\mathcal{F}}$ is called valid (or tautology), denoted by $\vdash \boldsymbol{\mathcal{F}}$,if there is no set

$\mathbf{P} = \{A_i:, i = 1, 2, \ldots, n\}$such that $\mathbf{P} = \{A_i:, i = 1, 2, \ldots, n\} \vdash \mathcal{F}$. In other words, awff $\mathcal{F}$ is called valid (or tautology) iff$\emptyset \vdash \mathcal{F}$,and $\mathcal{F}$ is called theorem in $(\boldsymbol{LPLS})$ by (CNDM).

Theorem 4.2.[Law of Excluded Middle]: Prove that $\mathcal{F} \coloneqq A \vee \neg A$ is a valid (or tautology).

proof. by 2-(CNDM)we want to show that$\vdash \mathcal{F} \coloneqq A \vee \neg A$.

Line #	wff	Reason	Dependency
1.	$\neg(A \vee \neg A)$	Ass	1.wff-ass.
2.	A	Ass	2.wff-ass.
3.	$A \vee \neg A$	$2, \vee -I$	3.wff.der.
4.	0 loop-dis-ass	$1,3, \neg - E$	4.wff.der.
5.	$\neg A$	$2,4, \neg - I$	{2,3,4}- {2}.
6.	$A \vee \neg A$	$5, \vee -I$	6.wff.der.
7.	0	$1,6,, \neg - E$	7.wff.der.
8.	$\neg\neg(A \vee \neg A)$	$1,7, \neg - I$	{1,2,3,4,5,6,7}-{1}.
9.	$A \vee \neg A$	$8, DN$	Discharge 4

The length of proof is equal to 9.So this proof is more efficiently. Hence the wff$\mathcal{F} \coloneqq A \vee \neg A$ is a valid or tautology■. Theorem 4. 3. Prove that $\mathcal{F} \coloneqq A \longrightarrow A$ is a valid.

proof. by 2-(CNDM):

Line #	wff	Reason	Dependency
1.	A	Ass	1.wff-ass.
2.	$\neg\neg A$	$1, (DN)^+$.	2.wff-der.
3.	A	$2, DN$	3.wff.der.
4.	$A \longrightarrow A$	$1,3, \longrightarrow -I$	{1,2,3}-{1}.

Hence the $\mathcal{F} \coloneqq A \longrightarrow A$ is a tautology. Note that the length of proof is equal to 4■.

Theorem 4.4.Prove that $\mathcal{F} \coloneqq (\neg A \longrightarrow A) \longrightarrow A$ is a valid.

proof. by 2-(CNDM):

Line #	wff	Reason	Dependency
1.	$\neg A \longrightarrow A$	Ass	1.wff-ass.
2.	$\neg A$	Ass	2.wff-ass.
3.	A	$1,2, \vee -I$	3.wff.der.
4.	0	$1,3, \neg - E$	4.wff.der.
5.	$\neg\neg A$	$2,4, \neg - I$	{2,3,4}- {2}.
6.	A	$5, DN$	6.wff.der.
7.	$(\neg A \longrightarrow A) \longrightarrow A$	$1,6,, \longrightarrow -I$	{1,2,3,4,5,6}-{1}.

Therefore, the wff $\mathcal{F} \coloneqq (\neg A \longrightarrow A) \longrightarrow A$ is a tautology and length of proof is 7■.

Definition 4.5. A wff $\mathcal{F}$ is calledcontradiction (or called un-satisfiability), if $\mathcal{F} \vdash \mathbf{0}$ $(or\ A \wedge \neg A)$.
In other word, awff $\mathcal{F}$ is called contradiction iff$\emptyset \vdash \neg\mathcal{F}$, that is the negate of $\mathcal{F}$ is a tautology.
Theorem 4.6.Prove that $\mathcal{F} \coloneqq (A \leftrightarrow \neg A)$ is a contradiction.

proof. by (CNDM) we desire to show that $\mathcal{F} \coloneqq (A \leftrightarrow \neg A) \vdash 0$.

Line #	wff	Reason	Dependency
1.	$(A \leftrightarrow \neg A)$	pre	1.wff-pre.
2.	$A \longrightarrow \neg A$	$1, \leftrightarrow -E$	2.wff-der.
3.	$\neg A \longrightarrow A$	$1, \leftrightarrow -E$	3.wff.der.
4.	A	Ass	4.wff.ass.
5.	$\neg A$	$2,4, \longrightarrow -E$	5. wff-der.
6.	0	$4,5, \neg - E$	6.wff.der.
7.	$\neg A$	$4,6,, \neg - I$	{4,5,6}-{4}.
8.	A	$3,7, \longrightarrow -E$	7.wff.der.
9.	$A \wedge \neg A = 0$	$7,8, \wedge -I$	8.wff.der.

Hence the wff$\mathcal{F} \coloneqq (A \leftrightarrow \neg A)$ is a contradiction and the length of proof is equal to 9■.

Theorem 4.7. Prove that $\mathcal{F} \coloneqq (A \vee \neg A) \longrightarrow (B \wedge \neg B)$ is a contradiction.

proof. by (CNDM) we need to show that $\mathcal{F} \coloneqq (A \vee \neg A) \longrightarrow (B \wedge \neg B) \vdash \mathbf{0}$ is a contradiction.

Line #	wff	Reason	Dependency
1.	$(A \vee \neg A) \longrightarrow (B \wedge \neg B)$	pre	1.wff-pre.
2.	A	Ass	2.wff-ass.
3.	$A \vee \neg A$	$2, \vee -I$	3.wff.der.
4.	$(B \wedge \neg B)$	$1,3, \longrightarrow -E$	4.wff.der.
5.	B	$4, \wedge -E$	5. wff-der.
6.	$\neg B$	$4, \neg - E$	6.wff.der.
7.	0	$5,6, \neg - E$	7.wff.der.
8.	$\neg A$	$5,7, \neg - I$	{2,3,4,5,6,7}-{2}.
9.	$A \vee \neg A$	$8, \vee -I$	9.wff.der.
10.	$B \wedge \neg B$	$1,9, \longrightarrow -E$	10.wff.der.
11.	B	$10, \wedge -E$	11.wff.der.
12.	$\neg B$	$10, \wedge -E$	11.wff.der.
13.	0	$11,12, \neg - E$	11.wff.der.

Since we get in last line a contradiction, therefore $\mathcal{F} \coloneqq (A \vee \neg A) \longrightarrow (B \wedge \neg B)$ is a contradiction■.

and consequently, $\boldsymbol{\mathcal{F}} \coloneqq (\boldsymbol{A} \vee \neg \boldsymbol{A}) \longrightarrow (\boldsymbol{B} \wedge \neg \boldsymbol{B}) \vdash \mathbf{0}$,with length proof is equal to 13.
Theorem 4.8. Prove that $\boldsymbol{\mathcal{F}} \coloneqq (\boldsymbol{A} \longrightarrow \boldsymbol{B}) \wedge (\boldsymbol{A} \wedge \neg \boldsymbol{B})$ is a contradiction.
proof. by (CNDM) we need to show that $\boldsymbol{\mathcal{F}} \coloneqq (\boldsymbol{A} \longrightarrow \boldsymbol{B}) \wedge (\boldsymbol{A} \wedge \neg \boldsymbol{B}) \vdash \mathbf{0}$ is a contradiction.

Line #	wff	Reason	Dependency
1.	$(A \longrightarrow B) \wedge (A \wedge \neg B)$	pre	1.wff-pre.
2.	$(A \longrightarrow B)$	$4, \wedge -E$	2.wff-der.
3.	$(A \wedge \neg B)$	$4, \wedge -E$	3.wff.der.
4.	A	$3, \wedge -E$	4.wff.der.
5.	$\neg B$	$3, \wedge -E$	5. wff-der.
6.	B	$2,4, \longrightarrow -E$	6.wff.der.
7.	0	$5,6, \neg - E$	7.wff.der.

So, we get in last line a contradiction, therefore $\mathcal{F} \coloneqq (A \longrightarrow B) \wedge (A \wedge \neg B)$ is a contradiction and the length of proof is equal to 7■.

5. Determines Relations between Propositions in (CNDM) for $(LPLS)$

Definition 5.1. Let A and B be two **wffs**, then A is said to be *logically implies* to B in the (CNDM) for $(LPLS)$, if $A \vdash B$.

Theorem 5.2. [TLI]: Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ and $\boldsymbol{D}$ be four wffs, 0 is a contradiction wff and 1 is a tautology wff, then:

1. $0 \vdash A \vdash 1$.	10. $(A \to B) \wedge (B \to C) \vdash A \to C$.
2. $A \vdash A \vee A$.	11. $(A \to B) \vdash (A \vee C) \to (B \vee C)$.
3. $A \wedge A \vdash A$.	12. $(A \to B) \vdash (A \wedge C) \to (B \wedge C)$.
4. $A \vdash 0 \vdash \neg A$.	13. $(A \to B) \vdash (B \to C) \to (A \to C)$.
5. $A \wedge (A \to B) \vdash B$.	14. $(A \to B) \vdash (B \to C) \to (A \to C)$.
6. $(A \to B) \wedge \neg A \vdash \neg A$.	15. $(A \to B) \wedge (C \to D) \vdash (A \wedge C) \to (B \wedge D)$.
7. $(A \vee B) \wedge \neg A \vdash B$.	16. $(A \to B) \wedge (C \to D) \vdash (\neg B \vee \neg D) \to (\neg A \vee \neg C)$.
8. $(A \vdash B) \to (A \wedge B)$.	17. $(A \to B) \wedge (C \to D) \vdash (\neg B \wedge \neg D) \to (\neg A \wedge \neg C)$.
9. $(A \leftrightarrow B) \wedge (B \leftrightarrow C) \vdash A \leftrightarrow C$.	18. $A \leftrightarrow \neg B \vdash \neg(A \leftrightarrow B)$.

Remark. We get in particular: $A \vdash B$ iff $A \Longrightarrow B$. We select 10,17 and 18as sample proof.

proof. (10): by (CNDM) we need to show that $(\boldsymbol{A} \to \boldsymbol{B}) \wedge (\boldsymbol{B} \to \boldsymbol{C}) \vdash \boldsymbol{A} \to \boldsymbol{C}$. (transitive relation of logical implies).

Line #	wff	Reason	Dependency
1.	$(A \longrightarrow B) \wedge (B \to C)$	pre	1.wff-pre.
2.	$(A \longrightarrow B)$	$1, \wedge -E$	2.wff-der.
3.	$(B \to C)$	$1, \wedge -E$	3.wff.der.
4.	A	Ass	4.wff.ass.
5.	B	$2,4 \longrightarrow -E$	5. wff-der.
6.	C	$3,5, \longrightarrow -E$	6.wff.der.
7.	$A \to C$	$4,6, \longrightarrow -I$	{4,5,6}-{4}.

The length of proof is equal to 7.

proof. (14):

Line #	wff	Reason	Dependency
1.	$A \longrightarrow B$	pre	1.wff-pre.
2.	$B \longrightarrow C$	Ass	2.wff.ass.
3.	A	Ass	3.wff.ass.
4.	B	$1,3 \longrightarrow -E$	4.wff.der.
5.	C	$2,4 \longrightarrow -E$	5. wff-der.
6.	$B \to C$	$4,5, \longrightarrow -I$	6. wff-der.
7.	$A \to C$	$3,5, \longrightarrow -I$	7.dis.ass.3.
8.	$(B \to C) \to (A \to C)$	$4,5, \longrightarrow -I$	8.dis.ass.3.

proof. (17):by (CNDM) we want to show that: $(A \rightarrow B) \wedge (C \rightarrow D) \vdash (\neg B \wedge \neg D) \rightarrow (\neg A \wedge \neg C)$.

Line #	wff	Reason	Dependency
1.	$(A \longrightarrow B) \wedge (C \rightarrow D)$	pre	1.wff-pre.
2.	$(\neg B \wedge \neg D)$	Ass	2.wff-ass.
3.	$(A \longrightarrow B)$	$1, \wedge -E$	3.wff.der.
4.	$(C \rightarrow D)$	$1, \wedge -E$	4.wff.der.
5.	A	Ass	5.wff-ass.
6.	B	$3,5 \longrightarrow -E$	6. wff-der.
7.	$\neg B$	$2, \wedge -E$	7.wff.der.
8.	0	$6,7 \neg - E$	8.wff.der.
9.	$\neg A$	$5,8 \neg - I$	{5,6,7,8}- {5}.
10.	C	Ass	10.wff.ass.
11.	D	$4,10, \longrightarrow -E$	11.wff.der.
12.	$\neg D$	$2, \wedge -E$	12.wff.der.
13.	0	$11,12 \neg - E$	13.wff.der.
14.	$\neg C$	$10,13 \neg - I$	{10,11,12,13}- {10}.
15.	$(\neg A \wedge \neg C)$	$9,14, \wedge -I$	15.wff.der.
16.	$(\neg B \wedge \neg D) \longrightarrow (\neg A \wedge \neg C)$	$\mathbf{2,15 \longrightarrow -I}$	{2, 3,…,15}- {2}.

Note that the length of proof is equal to 16.

proof. (18): by (CNDM) we want to show that: $A \leftrightarrow \neg B \vdash \neg(A \leftrightarrow B)$

Line #	wff	Reason	Dependency
1.	$A \leftrightarrow \neg B$	pre	1.wff-pre.
	$\neg\neg(A \leftrightarrow B)$	Ass	2.wff-ass.
3.	$(A \leftrightarrow B)$	$2, DN$	3.wff.der.
4.	$(A \longrightarrow B) \wedge (B \longrightarrow A)$	$3, \leftrightarrow -E$	4.wff.der.
5.	$A \longrightarrow B$	$4, \wedge -E$	5.wff-der.
6.	$B \longrightarrow A$	$4, \wedge -E$	6. wff-der.
7.	$(A \longrightarrow \neg B) \wedge (\neg B \longrightarrow A)$	$1, \leftrightarrow -E$	7.wff.der.
8.	$\neg B \longrightarrow A$	$7, \wedge -E$	8.wff.der.
9.	$A \longrightarrow \neg B$	$7, \wedge -E$	9.wff.der.
10.	A	Ass	10.wff.ass.
11.	B	$5,10, \longrightarrow -E$	11.wff.der.
12.	$\neg B$	$9,10, \longrightarrow -E$	12.wff.der.
13.	0	$11,12 \neg - E$	13.wff.der.
14.	$\neg A$	$11,12 \neg - I$	{10,11,12,13}- {10}.
15.	B	Ass	15.wff.ass.
16.	A	$\mathbf{6,15 \longrightarrow -E}$	16.wff.der.
17.	0	$\mathbf{14,16 \neg - E}$	17.wff.der.
18.	$\neg B$	$\mathbf{15,17 \neg - I}$	{15,16,17}- {15}.
19.	A	$\mathbf{8,18, \longrightarrow -E}$	19.wff.der.
20.	0	$\mathbf{14,19 \neg - E}$	20.wff.der.
21.	$\neg(A \leftrightarrow B)$	$\mathbf{2,20 \neg - I}$	{2,3, 4,…,20}- {2}.

Observation that the length of proof is equal to 21■.

Definition 5.3. Let A and B be two **wffs**, we say that A is a *provably equivalent to*B,if in (CNDM) for $(LPLS)$, if $A \vdash B$ $B \vdash A$ and denoted by $A \dashv\vdash B$.

Theorem 5.4. [Provably Equivalent] (TPE): Let A, B**and** C **be wffs, then:**

Provably equivalent	Name of provably equivalent
1. $A * B \dashv\vdash B * A$; $* = \wedge, \vee$ or $\leftrightarrow$	"Commutative- laws".
2. $A * (B * C) \dashv\vdash (A * B) * C$; $* = \wedge, \vee$ or $\leftrightarrow$	"Associative- laws ".
3. $A \vee (B \wedge C) \dashv\vdash (A \vee B) \wedge (A \vee C)$ and 4. $A \wedge (B \vee C) \dashv\vdash (A \wedge B) \vee (A \wedge C)$.	" Distributive- laws ".
5. $A \vee A \dashv\vdash A$ and $A \wedge A \dashv\vdash A$.	"Idempotent- laws ".
6. $\neg\neg A \dashv\vdash A$.	"Double negation".
7. $(A \vee 0) \dashv\vdash A$ and $(A \wedge 1) \dashv\vdash A$	
8. $A \rightarrow B \dashv\vdash \neg B \rightarrow A$ 9. $A \rightarrow \neg B \dashv\vdash B \rightarrow \neg A$	"contraposition-laws".
10. $A \rightarrow B \dashv\vdash \neg B \rightarrow \neg A$	
11. $A \rightarrow (B \rightarrow C) \dashv\vdash (A \wedge B) \rightarrow C$	"Exporation.law".
12. $A \wedge (A \vee B) \dashv\vdash A$ and $A \vee (A \wedge B) \dashv\vdash A$	" Absorption-law ".
13. $\neg(A \vee B) \dashv\vdash \neg A \wedge \neg B$ 14. $\neg(A \wedge B) \dashv\vdash \neg A \vee \neg B$. 15. $(A \vee B) \dashv\vdash \neg(\neg A \wedge \neg B)$. 16. $(A \wedge B) \dashv\vdash \neg(\neg A \vee \neg B)$. 17. $A \vee \neg B \dashv\vdash \neg(\neg A \wedge B)$ 18. $A \wedge \neg B \dashv\vdash \neg(\neg A \vee B)$	"Demorgan's-laws".
19. $A \longrightarrow B \dashv\vdash \neg A \vee B$.	

20. $A \longrightarrow B \dashv\vdash \neg(A \wedge \neg B)$. 21. $(A \vee B) \dashv\vdash (A \rightarrow B) \longrightarrow B$. 22. $(A \vee B) \dashv\vdash \neg A \rightarrow B$.	"Relations between logical connectives".
23. $A \leftrightarrow B \dashv\vdash (A \wedge B) \vee (\neg A \wedge \neg B)$	

Observation each wff of theorem provably equivalent deduced y first and second of category natural deduction for $(LPLS)$.In addition, we get in particular: $A \dashv\vdash B$ iff$A \Leftrightarrow B$.

Proof. (2) when $*=\vee$. First direction: $\boldsymbol{A} \vee (\boldsymbol{B} \vee \boldsymbol{C}) \vdash (\boldsymbol{A} \vee \boldsymbol{B}) \vee \boldsymbol{C}$.

Line #	wff	Reason	Dependency
1.	$A \vee (B \vee C)$	pre	1.wff-pre.
2.	A	Ass	2.wff-ass.
3.	$(A \vee B)$	$2,\vee -I$	3.wff.der.
4.	$(A \vee B) \vee C$	$3,\vee -I$	4.wff.der.
5.	$(B \vee C)$	Ass	5.wff-ass.
6.	B	Ass	6. wff-ass.
7.	$(A \vee B)$	$6,\vee -I$	7.wff.der.
8.	$(A \vee B) \vee C$	$7,\vee -I$	8.wff.der.
9.	C	Ass	9.wff.ass.
10.	$(A \vee B) \vee C$	$9,\vee -I$	10.wff.der.
11.	$(A \vee B) \vee C$	$5,6,,8,9,10,,\vee -E$	Dis by,$\vee -E$
12.	$(A \vee B) \vee C$	$1,2,4,5,11,\vee -E$	Dis by,$\vee -E$

By similar way, we get: $(\boldsymbol{A} \vee \boldsymbol{B}) \vee \boldsymbol{C} \vdash \boldsymbol{A} \vee (\boldsymbol{B} \vee \boldsymbol{C})$.Hence, $\boldsymbol{A} \vee (\boldsymbol{B} \vee \boldsymbol{C}) \dashv\vdash (\boldsymbol{A} \vee \boldsymbol{B}) \vee \boldsymbol{C}$ ■.

Proof. (4). First direction $A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$.

Line #	wff	Reason	Dependency
1.	$A \wedge (B \vee C)$	pre	1.wff-pre.
2.	A	$1,\wedge -E$	2.wff-der.
3.	$(B \vee C)$	$1,\wedge -E$	3.wff.der.
4.	B	Ass	4.wff.ass.
5.	$(A \wedge B)$	$2,4,\wedge -I$	5.wff-der.
6.	$(A \wedge B) \vee (A \wedge C)$	$5,\vee -I$	6. wff-der.
7.	C	Ass	7.wff.ass.
8.	$(A \wedge C)$	$2,7,\wedge -I$	8.wff.der.
9.	$(A \wedge B) \vee (A \wedge C)$	$8,\vee -I$	9.wff.der.
10.	$(A \wedge B) \vee (A \wedge C)$	$3,4,6,7,9,\vee -I$	Dis by ,$\vee -E$.

Conversely, to show that: $(\boldsymbol{A} \wedge \boldsymbol{B}) \vee (\boldsymbol{A} \wedge \boldsymbol{C}) \vdash \boldsymbol{A} \wedge (\boldsymbol{B} \vee \boldsymbol{C})$.

Line #	wff	Reason	Dependency
1.	$(A \wedge B) \vee (A \wedge C)$	pre	1.wff-pre.
2.	$(A \wedge B)$	Ass	2.wff-ass.
3.	A	$1,\wedge -E$	3.wff.der.
4.	B	$1,\wedge -E$	4.wff.der.
5.	$(B \vee C)$	$4,\vee -I$	5.wff-der.
6.	$A \wedge (B \vee C)$	$3,5 \wedge -I$	6. wff-der.
7.	$(A \wedge C)$	Ass	7.wff.ass.
8.	A	$7,\wedge -E$	8.wff.der.
9.	C	$7,\wedge -E$	9.wff.der.
10.	$(B \vee C)$	$9,\vee -I$	10.wff.der.
11.	$A \wedge (B \vee C)$	$8,10 \wedge -I$	10.wff.der.
12.	$A \wedge (B \vee C)$		Dis by ,$\vee -E$.

This complete the proof ■.

Definition 5.5. Let A and B be two **wffs**, we say that A is a *logical contradiction* to B,if $A \vdash \neg B$ and $B \vdash \neg A$.

Theorem 5.6.A is a *logical contradiction* to $\neg A$.

proof. To show that:$A \vdash \neg A$and$\neg A \vdash A$.

Line #	wff	Reason	Dependency
1.	A	pre	1.wff-pre.
2.	$\neg\neg A$	$1,(DN)^{+}$.	2.wff-der.
Another direction,			
1.	$\neg A$	pre	1.wff-pre.
2.	A	Ass	2.wff-ass.
3.	0	$1,2\neg - E$	3. wff-der.
4.	$\neg\neg A$	$2,3\neg - I$	4.dis.ass.2
5.	A	$4, DN$	5. wff-der.

6. Concepts of Consistency and Inconsistency in (CNDM) for ($LPLS$)

Definition 6.1. Let $\boldsymbol{\Psi} = \{\boldsymbol{A_i}: \boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \boldsymbol{3}, .., \boldsymbol{n}\}$ be a finite set of sequences of wffs in (CNDM) for $(\boldsymbol{LPLS}), \boldsymbol{\Psi}$ is called in*consistent* if there is a proof such that: $\boldsymbol{\Psi} = \{\boldsymbol{A_i}: \boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \boldsymbol{3}, .., \boldsymbol{n}\} \vdash \boldsymbol{0}$, In other word, if there is a wff$\boldsymbol{\mathcal{F}}$ such that both $\boldsymbol{\mathcal{F}}$ and $\neg\boldsymbol{\mathcal{F}}$ are derivable from the set of$\boldsymbol{\Psi}$,otherwise is called consistence set.

Theorem 6.2. Prove that the set of wffs$\boldsymbol{\Psi} = \{\boldsymbol{A} \leftrightarrow \boldsymbol{B}, \boldsymbol{B} \longrightarrow \neg\boldsymbol{A}, \boldsymbol{A}\}$**is inconsistent set.**

proof.

Line #	wff	Reason	Dependency
1.	$A \leftrightarrow B$	pre	1.wff-pre.
2.	$B \longrightarrow \neg A$	pre	2.wff-pre.
3.	A	pre	3.wff-pre.
4.	$(A \longrightarrow B) \land (B \longrightarrow A)$	$1, \leftrightarrow -E$	4.wff.der.
5.	$(A \longrightarrow B)$	$4, \land -E$	5.wff-der.
6.	B	$3,5 \longrightarrow -E$	6. wff-der.
7.	$\neg A$	$3,7 \longrightarrow -E$	7.wff.ass.
8.	0	$7, \neg - E$	8.wff.der.

Since $\Psi = \{A \leftrightarrow B, B \longrightarrow \neg A, A\} \vdash 0$,then $\Psi$is inconsistent set.

Theorem 6.3. Prove that the set of wffs $\boldsymbol{\Psi} = \{\boldsymbol{A} \longrightarrow (\boldsymbol{B} \land \neg\boldsymbol{B}), \boldsymbol{A}\}$**inconsistent set.**

proof.

Line #	wff	Reason	Dependency
1.	$A \longrightarrow (B \land \neg B)$	pre	1.wff-pre.
2.	A	pre	2.wff-pre.
3.	$B \land \neg B$	$1,2 \longrightarrow -E$	3.wff-pre.
4.	B	$4, \land -E$	4.wff.der.
5.	$\neg B$	$4, \land -E$	5.wff-der.
6.	0	$4,5 \neg - E$	6. wff-der.

Hence $\boldsymbol{\Psi} = \{\boldsymbol{A} \longrightarrow (\boldsymbol{B} \land \neg\boldsymbol{B}), \boldsymbol{A}\} \vdash \boldsymbol{0}$**, therefore wffs** $\boldsymbol{\Psi} = \{\boldsymbol{A} \longrightarrow (\boldsymbol{B} \land \neg\boldsymbol{B}), \boldsymbol{A}\}$ **is inconsistent set.**

Definition 6.4. Let $\boldsymbol{\Psi} = \{\boldsymbol{A_i}: \boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \boldsymbol{3}, .., \boldsymbol{n}\}$ be a finite set of sequences of wffsin (CNDM) for $(\boldsymbol{LPLS}), \boldsymbol{\Psi}$ is called *consistent*,$\boldsymbol{\Psi}$ is called if there is no a wff$\boldsymbol{A}$ such that both $\boldsymbol{A}$ and $\neg\boldsymbol{A}$ are derivable from the set of $\boldsymbol{\Psi}$. I.e. $\boldsymbol{\Psi} \nvdash \boldsymbol{A}$.and $\boldsymbol{\Psi} \nvdash \neg\boldsymbol{A}$, by different expression,$\boldsymbol{\Psi}$ is not inconsistent.

Remark

1. $A_i: i = 1,2,3, .., n \vdash A \longrightarrow B$ iff $A_i: i = 1,2,3, .., n, A \vdash B$.
2.$\boldsymbol{A_i}: \boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \boldsymbol{3}, .., \boldsymbol{n} \vdash \boldsymbol{B}iff\boldsymbol{A_i}: \boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \boldsymbol{3}, .., \boldsymbol{n}, \neg\boldsymbol{B} \vdash \boldsymbol{0}$.
3.$A_i: i = 1,2,3, .., n \vdash B$iff $\vdash \bigwedge_{i=1}^{n} A_i \longrightarrow B$.

Conflicts of Interest: *I declare that there are no conflict interests associated with this publication.*

Funding Statement: *There is no financial support or (fund) for this work*

Data availability statement: There is no data available for this work.

REFERENCES

1. A. Al-Odhari, Algebraizations of Propositional Logic and Monadic Logic, in Indian Journal of Advanced Mathematics,Volume-3 Issue-1, April 2023,pp12-19.
2. A. Al-Odhari, Existential and Universal Quantifier Operators on Boolean Algebras and Their Properties, I.J. of Mathematical Achieve, 12 (5) (2021), 37-44.
3. A. Al-Odhari, Novel of Truth-Tree for Propositional Logic System (NTTPLS), IJRAR, Vol. 9 (4) (2022), 43-56, https://www.ijrar.org/archive, http://doi.one/10.1729/Journal.32040.
4. A. Al-Odhari, Some Characteristics of Monadic and Simple Monadic Algebras, I.J. of Mathematical Achieve, 13 (4) (2022), 1-6.
5. A. Al-Odhari., Aspects of Language Monadic Predicate Logic System plus Identity (LMPL+I), Prospects for Applied Mathematics and data Analysis (PAMDA), Vol. 01, No. 01, PP. 31-51, 2023.
6. A. Al-Odhari., Features of Propositional Logic, Pure Mathematical Sciences, Vol. 10, 2021, no. 1, 35 – 44, HIKARI Ltd, www.m-ikari.com, https://doi.org/10.12988/pms.2021.91275
7. Arnold, and I. Guessarian, Mathematics for Computer Science. Prentice Hall Europe-Masson, 1996.
8. G. Boole, An investigation into laws of thought, 1854. www.gutenberg.org
9. G. Forbes, Modern Logic, Oxford University Press, Oxford, 1994.

10. G. Hamilton, Logic for Mathematicians, Cambridge University Press, Cambridge, 1978.
11. H. Kahane, Logic and Philosophy. Wadsworth, Inc. 1990.
12. H. Pospessel, Introduction to Logic Propositional Logic, Prentice-Hall, Inc.
13. M. Bergmann, J. Morr, J. Nelson, The Logic Book, Mc Graw-Hill, Companies, Inc., 1221.
14. P. A. Fejer, and D. Simovici, Mathematical, Foundations of Computer Science, Springer-Verlag New York, Inc, 1991. https://doi.org/10.1007/978-1-4612-3086-1
15. R. C. Jeffrey, Formal Logic: Its Scope and Limits, McGraw-Hill, New York (1967).
16. W. Miller, Introduction to Mathematical Logic, arXiv: math/ 9601203V1[math.LO]16(1996) Jan 1996.
17. W. V. Quine, Methods of Logic, Holt, Rinehart and Winston, Inc, 1959.U. Schoning, Logic for Computer Scientists, Birkhauser Boston Basel Berlin, 1989. https://doi.org/10.1007/978-0-8176-4763-6.
18. W. V. Quine, Philosophy of Logic, Harvard University Press, Cambridge, Massachusetts and London England, 1986.
19. Y. I. Manin, A Course of Mathematical Logic for Mathematicians, Springer, New York, Dordrecht, Heidelberg, London, 2010. https://doi.org/10.1007/978-1-4419-0615-1

****