Available Online at http://www.journalajst.com

ASIAN JOURNAL OF SCIENCE AND TECHNOLOGY

ISSN: 0976-3376

Asian Journal of Science and Technology
Vol. 13, Issue, 12, pp.12350-12358, December, 2022

RESEARCH ARTICLE

ON THE P – VERTEX SPANNING SUBTREE POLYTOPE OF A GRAPH

[1]*Mamane Souleye Ibrahim and [2]Belko Soumana Boubacar

[1]Department of Mathematics and Computer Science, Abdou Moumouni University, Niamey, Niger
[2]Department of Mathematics and Computer Science, Abdou Moumouni University, Niamey, Niger

ARTICLE INFO

Article History:
Received 09th September, 2022
Received in revised form
10th October, 2022
Accepted 15th November, 2022
Published online 30th December, 2022

Keywords:
graph, algorithm, valid inequality, facet, polytope.

ABSTRACT

In this paper, given an undirected graph $G = (V,E)$, with $|V| = n$, we introduce a new integer linear description of the polytope $P_T(G)$ of p−vertex spanning subtrees of G. A p−vertex spanning subtree is a subtree that spans $p < n$ vertices of G. Unlike existing linear descriptions of such a polytope, ours is only defined on the space of variables associated with edges of G and is based on well known partition inequalities. After, we address constructive algorithms generating p − vertex spanning subtrees that incidence vectors are affinely independent to determine the dimension of $P_T(G)$and to show the facetness of trivial inequalities$x_e \geq 0$ and $x_e \leq 1$.

Citation: ***Mamane Souleye Ibrahim and Belko Soumana Boubacar.*** **2022.** "On the p − vertex Spanning Subtree Polytope of a Graph", *Asian Journal of Science and Technology*, 13, (12), 12350-12358.

INTRODUCTION

Given the undirected graph $G = (V,E)$, where V is the set of vertices, E the set of edges and such that $|V| = n$. A p − vertex spanning sub tree of G is a tree of G that spans $p < n$ vertices. Consider the collection θ of all p−vertex spanning subtrees of G. For some $T \in \theta$, the incidence vector x of the p− vertex spanning sub tree is defined as follow:

For all $e \in E$, $x_e = \begin{cases} 1 & e \in T \\ 0 & e \notin T \end{cases}$
for some T in θ.

Assume that each edge, $e \in E$, has a weight $w(e) \in R_+$, the p−vertex spanning subtree problem (p-VSSP for short) consists, given p, to find a p − vertex spanning subtree T^* with minimum total weight. The total weight of a tree is the sum of the weight of its edges. We denote by p-VSSP (G), the convex hull of incidence vectors of p − vertex spanning subtrees of G. Formally, we have:

$$p\text{-}VSSP(G) = conv\,\{x_e \in \{0,1\}^E : \mathit{for\ all}\ T \in \theta\}$$

That is *p-VSSP* can be defined as:

Minimize $\{wx : x \in p\text{-}VSSP(G)\}$

p-VSSP is NP-hard. Indeed, in Fischetti *et al*., (1994), authors show that the Steiner tree, known to be strongly NP-hard, (see, Garey& Johnson, (1979)), can be reduced to p-VSSP. The *p-VSSP* has various application domains among which we cite the oil-field leasing (Hamacher & Joernsten, (1992)), facility layout (Foulds&Hamacher, (1992)), open pit mining (Philpott&Wormald, (1997)), telecommunications (Garg&Hochbaum, (1996)). For other application examples of p−V SSP, one can refer to (Blum &Ehrgott(2003)). In literature, the p-vertex spanning subtree problem is also called the *k*-cardinality tree problem. Several studies has been conducted in the literature on the subject. The first integer linear program (ILP) formulation of the p−vertex spanning subtree problem is due to Fischetti *et al*. Fischetti *et al*. (1994). To define the model, au- thors consider two types of binary variables, say xe and yv, associated to the edge e and the vertex v of the graph, respectively.

Corresponding author: **Mamane Souleye Ibrahim,**
Department of Mathematics and Computer Science, Abdou Moumouni University, Niamey, Niger.

They also discuss the facial structure of the problem polytope. Note that Fischetti *et al*., (1994) p− vertex spanning subtree problem linear formulation has been implemented by Ehrgott *et al*., Ehrgott *et al*. (1994). Maculan *et al*. Maculan *et al*., (2003) present a flow based linear formulation of the p − vertex spanning subtree problem. In their formulation, they first transform the undirected graph into a digraph and add an arti- ficial vertex which may play the role of a root vertex. A vertex in a digraph, say r, is called a root vertex if there exists at least a simple path between the root vertex r and all other vertices of the digraph. After, in addition to binary variables associated to vertices and edges of the graph, they also consider flow continuous positive variables $f_{u,v}^{w} \geq 0$ that define the flow that passes by the arc (u, v) and is destined to the sink vertex w. In Chimani *et al*.,(2008) Chimani *et al*. (2008), to efficiently solve the problem using a Branch and Cut algorithm, authors considerwhat they call the k−cardinality arborescence problem. Indeed, as in Maculan *et al*., (2003), authors also transform the undirected graph that represents an instance of the p−vertex spanning subtree problem into a directedinstance and create an artificial root vertex. Chimani *et al*. Chimani *et al*. (2008) show that their formulationis equivalent to the one introduced by Fischetti *et al*. Fischetti *et al*. (1994) from the polyhedral point of view. Another linear formulation of *p-VSSP* is due to Garg (1996) Garg, (1996). He presents a formulation based on undirected cuts. Similarly to linear formulations of *p-VSSP* of Maculan *et al*. (2003) Maculan *et al*., (2003), andChimani *et al*. (2008) Chimani *et al*. (2008), its formulation also uses the concept of root vertex. However, here the root vertex, instead to be an artificial vertex, is selected among vertices of the consider graph. On the other hand, p − V SSP can be classified in the wide field of network design problem. Grotschel and Monma Grotschel & Monma., (1990) introduce a general integer linear model for the problem of designing minimum cost survivable networks. This general model includes special cases as the minimum spanning tree problemobtained by fixed what is called vertex connectivity type r_u to 1 for all $u \in V$, the Steiner tree problem obtained,given a subset S of vertices, by fixed all $u \in S$, $r_u = 0$ and for all $u \in V \setminus S$, $r_u = 1$ and the minimum cost k-edgeconnected and k-node connected network design problems by fixed$r_u = k$ for all $u \in V$. For more details withrespect to network design model based on the concept of vertex connectivity concept, see also Grotschel *et al*.,(1992)Grotschel *et al*., (1995). For an interesting survey concerning network design problem, on can refer to the paper of Hervé Kerivin and Ridha Mahjoub (2005). In this paper, we introduce a new integer linear description of the polytope of p−vertex spanning subtrees of G, where $p < n$. Unlike all other linear formulations of *p-VSSP*, ours is defined only on the space of variables associated with edges of the graph G. We recall that existing linear models of the problem take into account at least space defined by both variables associated with edges and vertices of the graph, (see, Fischetti *et al*. (1994) Garg, (1996) Maculan *et al*., (2003) and Chimani *et al*. (2008)). As we will see such a polytope is mainly based on partition inequalities. We recall that Grotschel and Monma (1990) show that partition inequalities combined with trivial inequalities suffice to describe the spanning tree polytope. After, resorting to constructive algorithms that generate p-vertex spanning subtrees, we discuss the facetness of inequalities that define the subtreepolytope. The paper is organized as follows. In section 2, we introduce a new integer linear program of *p-VSSP*. Such a ILP-program is defined on the space of variables associated with the edges of the graph. In Section 3, we discuss the dimension and defining facet of p−vertex spanning subtreepolytope. For this purpose, we devise constructive algorithms to generate p-subtrees with affinely independent incidence vectors. In the rest of this section, we give further definitions and notations. Throughout the paper, we deal with the

complete undirected graph $G = (V,E)$, with $V = \{1, 2, . . . , n\}$, $E = \{e_{k,l} = (k, l), 1 \leq k \leq n - 1, 2 \leq l \leq n\}$.

That is, we have $|E| = m = n(n-1)$.

We denote an edge as a pair of vertices. Let$\boldsymbol{\tau_i}$be the incidence vector ofthe p−vertex spanning subtree T_i.

$E(Ti)$ is the edge set of the subtreeT_i and $|E(T_i)|$is the number of edges of T_i. Recall that vectors $\tau_1, \tau_2, . . . , \tau_q$ are said to be affinely independent, if there exists some coefficientsλ_i, $1 \leq i \leq q$ such that the unique solution of systems$\sum_{i=0}^{q} \lambda_i \tau_i = 0$ and $\sum_{i=0}^{q} \lambda_i = 0$ is$\lambda_i = 0, i = 1, . . . , q$. Inthesequel, we consider a partition $\pi = (V_1, V_2, . . . ,V_r)$ of V such that $1 \leq |V_j| \leq p-1, j = 1, . . . , r$. That is, we have$V_1 \cup V_2 \cup . . . \cup V_r = V$ and $V_j \cap V_{j'} = \emptyset$, $\forall j, j' \in \{1, . . . , r\}$. Given a partition $\pi = (V_1, V_2, . . . ,V_r)$, we denote by$\delta((V_1, V_2, . . . , V_r))$ the set of edges with endpoints in two different components. By $\delta([V_j : V_{j'}])$, we mean the edgeset having one of its endpoint in V_j and the other in $V_{j'}$. Given two components, V_j and $V_{j'}$of a partition π, by$e_{k,l}^{j,j'}$or $f_{k,l}^{j,j'}$, we define the edge (k, l) such that vertices k and l belong to components V_j and $V_{j'}$, respectively.$e_{k,l}^{j}$or$f_{k,l}^{j}$ denotes the edge (k, l) with both vertices k and l belonging to the component V_j. We denote by $E(V_i)$the set of edge having both its endpoints in V_i and $G[V_i] = (V_i,E(V_i))$ the subgraph induced by V_i. The degree$d_G(u)$ of a given vertex u of G is the number of edges having the vertex u as endpoint. We call all leaf vertices$u$-rooted p−vertex subtree T in G, a p−vertex subtree having the vertex u as a root such that $d_G(u)= p-1$ and$d_G(v) = 1$, for all others vertices v of T. As an example, subtreesT_1, T_2 and T_3 depicted in Figure 1 are all leafvertices 1−rooted p−vertex spanning subtrees, with the vertex 1 as a root and $p = 4$.

The Subtreepolytope

A new ILP for p−VSSP: Consider the complete undirected graph $G = (V,E)$. Given a p-vertex spanning subtree$T$ of the convex hull *p-VSSP(G)*, its incidence vector x satisfies the following inequalities:

$$x(E) = p - 1, \qquad (1)$$

$$x(\delta(\pi)) \geq 1, \ \forall \pi, \qquad (2)$$

$x_e \in \{0, 1\}, e \in E.$ (3)

Where π is a partition of the vertex set V defined as above.

Constraint (1) guarantees the cardinality condition. Indeed, p − vertex spanning subtrees may contain (p − 1) edges. Constraints (2) are partition inequalities that permit simultaneously to eliminate cycles in any solution of *p-VSSP(G)* and to make such a solution connected. Constraints of type (3) are integrality constraints. In Grotschel & Monma., (1990), authors showed that inequalities (1), (2) and (3) suffice to describe the spanning treepolytope. Note that in this case $p = n$ and the subsets V_i, $i = 1, \dots, r$ that form the partition are such that

$1 \le |V_i| \le n - 1.$

Theorem 1. Given $x \in \{0, 1\}^E$satisfying constraints (1) and (2), then $E(T) = \{e \in E : x_e = 1\}$is an edge set of a p−vertex spanning subtree T.

Proof. From the definition, a p−vertex spanning subtree$T$ is an acyclic connected subraph of G having an edge set cardinality,$|E(T)|$, equals to $p-1$. Assume that the edge set $E(T)$ do not form a p−vertex spanning subtree T. This implies that T satisfies at least one of the following cases

• $|E(T)| = p - 1$. In this case, constraint (1) is violated.
• T contains at least a cycle, is connected and is such that$|E(T)| = p - 1$. W.l.o.g., assume that T contains an unique cycle C, with$|E(C)| = k$, where $E(C)$ is the edge set of thecycle C. If $k > p - 1$, constraint (1) is violated. If $k = p - 1$, there exists a partition π that induces aconstraint of type (2) violated by the incidence vector of T. Indeed, such a partition can be constructedsuch that all vertices of the cycle C belong to one of its component. So, consider that $k < p-1$ implying that T contains $(p-1-k)$ other edges that do not belong to the cycleC. By hypothesis, as T is connected and contains the unique cycle C, each edge of T that do not belong to C is incident to a vertex of $V \setminus V(C)$. Therefore, with the cycle C, we create a connected structure T with $(p-1)$ vertices and $(p-1)$ edges. We can then construct a partition π with a component that contains all vertices and edges of T. It then follows that constraint (2) corresponding to such a partition is violated by the incidence vector of the structure T.
•T is not connected, is acyclic and is such that$|E(T)| = p - 1$. W.l.o.g., assume that T has 2 connected components, say C_1 and C_2. It's obvious that the partition π, with 2 distinct components that contain C1 and C2 respectively, induces a constraint of type (2) violated by the incidence vector of T. We conclude that constraints (1)-(3) issatisfied by all solutions of$p-V\,SSP$.

If we replace the integrality constraints (3) by the inequalities

$x_e \ge 0, \forall e \in E,$ (4)

$x_e \le 1, \forall e \in E,$ (5)

we get the LP-relaxation of the ILP (1)-(3). We denote by $P_T(G)$, the polytope induced by constraints (1), (2) and (4)-(5). We then have $p-VSSP(G) \subset P_T(G)$. Note also that the polytope$P_T(G)$ is included in the affine space defined by $\{x \in \{0, 1\}^E : x(E) = p - 1\}$.

Dimension and facets of the p−VSSP polytope

Some technical lemmas: In what follows, we give some technical lemmas which will be useful in the proof of results of this section.

Lemma 1. From the set $Q_\tau = \{\tau_1\}$ where τ_1 is the incidence vector of a p−vertex spanning subtreeT_1, by sequentially setting $Q_\tau := Q_\tau \cup \{\tau_i\}$such that the p−vertex spanning subtree T_i contains an edge e that is not contained by any subtree $T_{i'}$ having its corresponding incidence vector in Q_τ ($\tau_{i'} \in Q_\tau$), then we construct a set Q_τ that elements are affinely independent.

Proof. Assume that $Q_\tau = \{\tau_1, \tau_2, \dots, \tau_q\}$is a set of affinely independent subtree incidence vectors and con- sider the subtreeT_{q+1}, (with τ_{q+1} as incidence vector), that contains the edge e with the condition that $e \notin E(T_i)$, $i = 1, \dots, q$. By applying the definition of affine independence with respect to the set $Q_\tau \cup \{\tau_{q+1}\}$, one can write $\sum_{i=1}^{q} \lambda_i + \lambda_{q+1} = 0$. As $\sum_{i=0}^{q} \lambda_i = 0$, we deduce that $\lambda_{q+1} = 0$. On the other hand, the vectors $\tau_1, \tau_2, \dots, \tau_q$ representing the subtrees$T_1, T_2, \dots, T_q$ are affinely independent, this proves that the incidences vectors$\tau_1, \tau_2, \dots, \tau_q, \tau_{q+1}$ are affinely independent.

Lemma 2. Consider a set $\{\tau_1, \tau_2, \dots, \tau_q\}$ of affinely independent incidence vectors of p−vertex spanning subtrees T_i, $i = 1, \dots, q$, constructed according to Lemma 1, that all pass through an edge, say $e_{1,2} = (1, 2)$. If the p−vertex spanning subtrees $T_{q+1}, \dots, T_{q+l}$, (with incidence vectors $\tau_{q+1}, \dots, \tau_{q+l}$) contain the edge $e_{1,2}$ and $(p - 3)$ otheredges $e_{1,k} = (1, k)$, $k \in \{3, \dots, p - 1\}$ such that for each subtreeT_{q+j} , $j \in \{1, \dots, l\}$, there exists an edge

$e_{1,k} = (1, k)$, $3 \le k \le p-1$ with $e_{1,k} = (1, k) \notin E(T_{q+j})$ and $e_{1,k} \in E(T_{q+j'})$, $j' \in \{1, \dots, l\} \setminus \{j\}$. Then incidence vectors$\tau_1, \tau_2, \dots, \tau_q, \tau_{q+1}$ are affinely independent.

Proof. As vectors $\tau_1, \tau_2, \ldots, \tau_q$ are affinely independent according to Lemma 1 and the fact that all p−vertex spanning subtrees pass through the edge $e_{1,2} = (1, 2)$, applying the affine independence definition, we have $\sum_{j=1}^{q} \lambda_j + \sum_{j=1}^{l} \lambda_{q+j} = 0$.On the other hand, each subtreeT_{q+j} is such that there exists an edge $e_{1,k} = (1, k)$, $k \in \{3, \ldots, p-1\}$contained by all subtrees$T_{q+j'}$, $j' \in \{1, \ldots, l\} \setminus \{j\}$except the subtree$T_{q+j}$.

So, we can write $(p-3)$ equations of the form $\sum_{j' \neq j} \lambda_{q+j'} + \sum_{j=1}^{q} \lambda_j = 0$.This finally implies that$\tau_{q+j} = 0$, $j = \{1, \ldots, l\}$ and shows that vectors $\tau_1, \tau_2, \ldots, \tau_q, \tau_{q+1}, \ldots, \tau_{q+l}$ are affinely independent.

Lemma 3. Consider a set $\{\tau_1, \tau_2, \ldots, \tau_q\}$of affinely independent incidence vectors of p−vertex spanning subtreesT_i, $i = 1, \ldots, q$, constructed according to Lemma 1, that all pass through an edge, say $e_{1,2} = (1, 2)$. Let T_{q+1} (withτ_{q+1} as incidence vector) be a p−vertex spanning subtree that do not pass by $e_{1,2}$. Then the incidence vectors$\tau_1, \tau_2, \ldots, \tau_q, \tau_{q+1}$ are affinely independent.

Proof. Applying the definition of affine independence, as$\sum_{j=1}^{q+1} \lambda_j = 0$ and the fact that all subtrees pass by the edge$e_{1,2}$, except the p−vertex spanning subtreeT_{q+1}, we deduce that$\sum_{j=1}^{q} \lambda_j = 0$ a implying that $\tau_{q+1} = 0$. That shows that the incidence vectors $\tau_1, \tau_2, \ldots, \tau_q, \tau_{q+1}$ are affinely independent.

Dimension of $P_T(G)$: In this subsection, to determine the dimension of the polytope$P_T(G)$, according to above lemmas, we present an algorithm that constructs (m) p − vertex spanning subtrees that incidence vectors are affinely independent. First, Algorithm 1 below constructs ($m-1$) p−vertex spanning subtrees that corresponding incidence vectors areaffinely independent and that all pass by a given edge, say $e_{1,2} = (1, 2)$. After, we join to these ($m-1$) subtrees a latter subtree that do not pass through the edge $e_{1,2} = (1, 2)$. For this, consider the 1−rooted p−vertex spanning subtreeT_1, (with incidence vector τ_1), that contains the edges $e_{1,2}, e_{1,3}, \ldots, e_{1,p}$. All edges $e_{1,k} = (1, k)$, $k = 2, \ldots, p$ share the root vertex 1 as endpoint. As $|E(T1)| = p-1$, T_1 do not pass through $m-(p-1) = m-p+1$ edges of G.

Let

$f_{1,j} = (1, j), j = p+1, \ldots, n$

and

$f_{k,l} = (k, l), 2 \leq k \leq n-1, 3 \leq l \leq n$, with $k < l$

be these (m − p + 1) edges. From T_1, we contruct the subtrees T_i, $i = 2, \ldots, m$that incidence vectors are, with the one of T_1, affinely independent. This is detailed in the following constructive procedure.

Algorithm 1: Computation of subtrees with affinely independent vectors

```
Data: The complete graph G = (V, E) and the subtree T_1, with E(T_1) = {e_{1,2}, e_{1,3}, ..., e_{1,p}}.
Result: Set of m subtrees with affinely independent incidence vectors.
begin
    // Note that T_1 is such that E(T_1) = {e_{1,2}, e_{1,3}, ..., e_{1,p}}
    i ← 2
    for k ← p+1 To n do
        Construct T_i such that E(T_i) = (E(T_1) \ {e_{1,p}}) ∪ {f_{1,k}}
        i ← i + 1
    end
    i ← n − p + 2
    for k ← 2 To p − 1 do
        for l ← k + 1 To p do
            Construct T_i such that E(T_i) = (E(T_1) \ {e_{1,l}}) ∪ {f_{k,l}}
            i ← i + 1
        end
        for l ← p + 1 To n do
            Construct T_i such that E(T_i) = (E(T_1) \ {e_{1,p}}) ∪ {f_{k,l}}
            i ← i + 1
        end
    end
end
```

Continue …

```
20 i ← (n − p)(p − 1) + (p−1)(p−2)/2 + 2;
21 for k ← p To n − 1 do
22 |   if k ← p then
23 |   |   for l ← p + 1 To n do
24 |   |   |   Construct T_i such that E(T_i) = (E(T_1) \ {e_{1,p−1}}) ∪ {f_{kl}};
25 |   |   |   i ← i + 1;
26 |   |   end
27 |   end
28 |   else
29 |   |   i ← p(n − p) + (p−1)(p−2)/2 + 2;
30 |   |   for l ← k + 1 To n do
31 |   |   |   Construct T_i such that E(T_i) = (E(T_1) \ {e_{1,p−1}, e_{1,p}}) ∪ {f_{1,k}, f_{k,l}};
32 |   |   |   i ← i + 1;
33 |   |   end
34 |   end
35 end
36 i ← m − p + 3;
37 for k ← 3 To p − 1 do
38 |   Construct T_i such that E(T_i) = (E(T_1) \ {e_{1,k}}) ∪ {f_{1,p+1}};
39 |   i ← i + 1;
40 end
41 i ← m;
42 Construct T_m such that E(T_m) = (E(T_1) \ {e_{1,2}}) ∪ {f_{1,p+1}};
43 𝒯 ← {T_1, . . . , T_m};
44 return 𝒯;
```

Theorem 2. Algorithm 1 constructs (m) p − vertex spanning subtrees that incidence vectors are affinelyinde-pendent.

Proof. By Lemma 1, steps 2-35 of Algorithm 1 construct ($m-p+2$) p−vertex spanning subtrees that all contain the edge $e_{1,2} = (1, 2)$ such that its incidence vectors τ_i, $i = 1, \ldots, m - p + 2$ are affinely independent. Indeed, in the construction processus, at each step, the current subtree T_i, $i = 2, \ldots, m - p + 2$, includes an edge that do not belong to any previously constructed subtreeT_j , $j = 1, \ldots, i - 1$. Such edges are represented by dashed arcs in Figures 1, 2 and 3 below for $n = 6$ and $p = 4$. According to Lemma 2, by applying steps 36-40 of Algorithm 1, in addition to the first $m-p+2$ already constructedsubtrees, we constructs $(p-3)$ other subtrees such that we obtain $(m-1)$ p−vertex spanning subtrees$T_1, \ldots, T_{m-1}$that incidence vectors $\tau_1, \ldots, \tau_{m-1}$ are affinely independent, see Figure 4 in the example described below. Steps 41 and 42 construct a p−vertex spanning subtreeT_m (withτ_m as incidence vector) that do not pass through the edge $e_{1,2} = (1, 2)$, (see Figure 4). As all other $(m - 1)$subtrees contain the edge $e_{1,2}$ and its corresponding vectors are affinely independent, by Lemma 3, vectors $\tau_1, \ldots, \tau_m$ are affinely independent showing that the dimension of $P_T(K_n)$ is equal to $m - 1$. This completes the proof. In the following illustrative example, we apply Algorithm 1 on the complete graph $G = (V,E)$ with $n = 6$ and$p = 4$ to generate (m) p − vertex spanning subtrees with affinely independent incidence vectors.

Note that

$m = n(n - 1)/2 = 15$.

Example 1. Consider the complete graph G with $V = \{1, 2, \ldots, 6\}$, $E = \{(u, v) : 1 \le u \le 5, 2 \le v \le 6, u < v\}$, $n = 6$, $p = 4$, $e_{1,2} = (1, 2)$ and T_1 is such that $E(T_1) = \{(1, 2), (1, 3), (1, 4)\}$. Figures 1-4 show the p−vertex subtrees constructed by applying the steps of the above constructive procedure that incidence vectors are affinely independent.

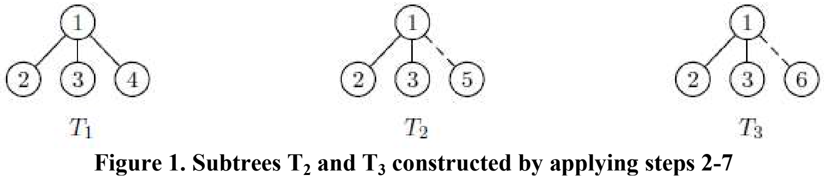

Figure 1. Subtrees T_2 and T_3 constructed by applying steps 2-7

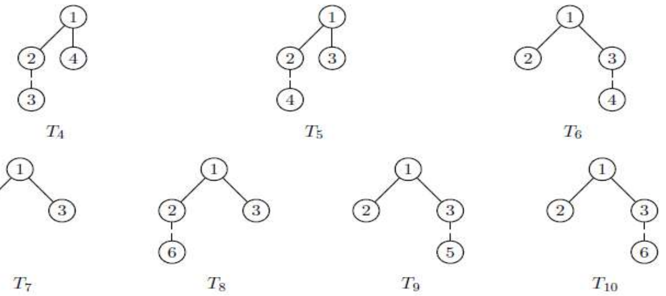

Figure 2. subtrees t4 –t10 constructed by apolying steps 8-19

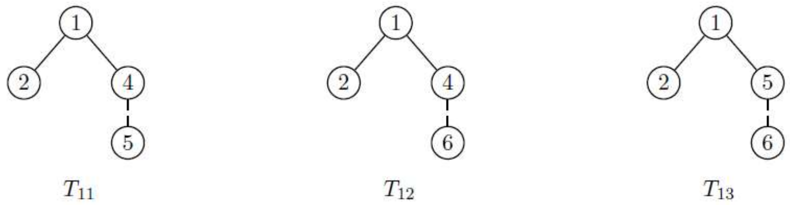

Figure 3. Subtrees T_{11}, T_{12} and T13 constructed by applying Steps 20-35 of Algorithm 1

Figure 4. Subtrees T_{14} and T_{15} displayed respectively by Steps 36-40 and 41-42

Theorem 3. The dimension of $P_T(G)$ is equal to $m - 1$.

Proof. By virtue of Theorem 2, it's possible to create (m) p − vertex spanning subtreesT_i, $i = 1, \ldots, m$, with τ_i, $i = 1, \ldots, m$ as incidence vectors such that vectors $\tau_1, \tau_2, \ldots, \tau_m$ are affinely independent. This completes the proof.

Facetness of trivial constraints: Consider the LP-relaxation of $p - VSSP$ defined in the previous section. As in the previous theorem, we give an algorithm that constructs ($m-1$) p−vertex spanning subtrees that corre-sponding incidence vectors are affinely independent and satisfy a valid inequality of type (4) with equality. Recallthat every p−vertex spanning subtree that incidence vector satisfies a valid inequality of type (4), say $x(e_{1,2}) \geq 0$,with equality do not contain the edge $e_{1,2} = (1, 2)$. For this, consider the 1−rooted p−vertex spanning subtreeT_1', (with incidence vector τ_1'), that contains the edges $e_{1,3}, e_{1,4}, \ldots, e_{1,p+1}$, with $e_{1,k} = (1, k), k = 3, \ldots, p + 1$.

```
Algorithm 2: Computation of subtrees with affinely independent vectors
Data: The graph G = (V, E) and the subtree T'_1, with E(T'_1) = {e_{1,3}, ..., e_{1,p+1}}.
Result: Set of (m − 1) subtrees with affinely independent incidence vectors that do not pass by e_{1,2}
begin
    i ← 2
    for k ← p + 2 To n do
        Construct T'_i such that E(T'_i) = (E(T'_1) \ {e_{1,p+1}}) ∪ {f_{1,k}}
        i ← i + 1
    end
    i ← n − p + 1
    for k ← 3 To n do
        if k ≤ p then
            Construct T'_i such that E(T'_i) = (E(T'_1) \ {e_{1,p+1}}) ∪ {f_{2,k}}
            i ← i + 1
        end
        if k ← p + 1 then
            Construct T'_{n−1} such that E(T'_{n−1}) = (E(T'_1) \ {e_{1,p}}) ∪ {f_{2,p+1}}
            i ← i + 1
        end
        else
            Construct T'_i such that E(T'_i) = (E(T'_1) \ {e_{1,p}, e_{1,p+1}}) ∪ {f_{1,k}, f_{2,k}}
            i ← i + 1
        end
    end
end
i ← 2n − p − 1;
for k ← 3 To p do
    for l ← k + 1 To p + 1 do
        Construct T'_i such that E(T'_i) = (E(T'_1) \ {e_{1,l}}) ∪ {f_{k,l}};
        i ← i + 1;
    end
    for l ← p + 2 To n do
        Construct T'_i such that E(T'_i) = (E(T'_1) \ {e_{1,p+1}}) ∪ {f_{k,l}};
        i ← i + 1;
    end
end
i ← p(n − 1) − p(p+1)/2 + 2;
k ← p + 1;
for l ← p + 2 To n do
    Construct T'_i such that E(T'_i) = (E(T'_1) \ {e_{1,p}}) ∪ {f_{k,l}};
    i ← i + 1;
end
i ← (n + 1) + p(n − 2) − p(p+1)/2;
for k ← p + 2 To n − 1 do
    for l ← k + 1 To n do
        Construct T'_i such that E(T'_i) = (E(T'_1) \ {e_p, e_{p+1}}) ∪ {f_{1,k}, f_{k,l}};
        i ← i + 1;
    end
end
```

All edges $e_{1,k} = (1, k)$, $k = 3, \ldots, p + 1$ share the root vertex 1 as endpoint. Note that, in addition to the edge $e_{1,2}$, T_1' do not contain $m - (p - 2 + 1) - 1 = m - p$ edges of G. Let

$f_{1,j} = (1, j), j = p + 2, \ldots, n$

and

$f_{k,l} = (k, l), 2 \leq k \leq n - 1, k + 1 \leq l \leq n$ be these edges.

From T_1', we contruct the subtrees T_i', $i = 2, \ldots, m - 1$ as follow:

Proof. By Lemma 1, Steps 2-46 of Algorithm 2 construct $(m - p + 1)$ p−vertex spanning subtrees, T'_i, $i = 1, \ldots, m - p + 1$, that do not contain the edge $e_{1,2} = (1, 2)$, but all contains the edge $e_{1,3} = (1, 3)$ such that its incidence vectors τ_i', $i = 1, \ldots, m-p+1$ are affinely independent. Indeed, in the construction processus, at each step, the current subtree$T_{i'}$, $i = 2, \ldots .m - p + 1$ includes an edge that do not belong to previuosly constructed subtrees T_1', …, T'_{i-1}. Such edges are the ones represented by dashed lines, (see Figures 5-7 below for $n = 6$ and $p = 4$). Therefore, by applying Steps 47-50 of the algorithm, we add to the first $m - p + 1$ already constructed subtrees, $(p - 3)$ subtrees$T'_{m-p+2}, \ldots, T'_{m-2}$. By Lemma 2, its incidence vectors are affinely independent. Notethat such subtrees also all contains the edge $e_{1,3} = (1, 3)$. At the end, by Lemma 3, applying Step 51, we jointo these $(m - 2)$ subtrees, the subtreeT'_{m-1} that do not pass through the edge $e_{1,3}$ (see, Figure 8). Thus, the vectors$\tau \cdot 1, \ldots, \tau'_{m-1}$ are affinely independents.

Example 2. Consider the graph $G=(V,E)$ with $V = \{1, 2, \ldots, 6\}$, $E = \{(u, v) : 1 \leq u \leq 5, 2 \leq v \leq 6, u < v\}$, $n = 6$, $p = 4$, $e_{1,2} = (1, 2)$ and T'_1 is such that $E(T'_1) = \{(1, 3), (1, 4), (1, 5)\}$. Figures below shows the p−vertex subtreesconstructed by applying the above constructive algorithm that incidence vectors are affinely independent.

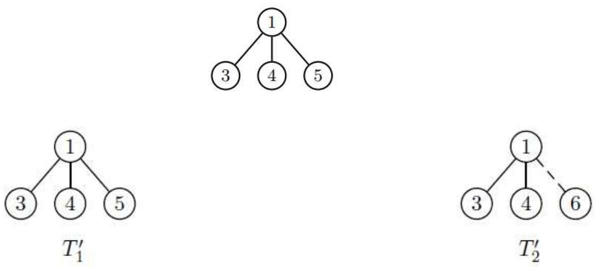

Figure 5. Subtree T_2' constructed by applying steps 2-6 of Algorithm 2

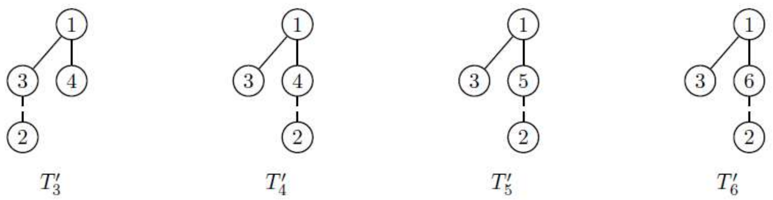

Figure 6: Subtrees $T_{3'}$ - T6′ constructed by applying steps 7-22 of Algorithm 2

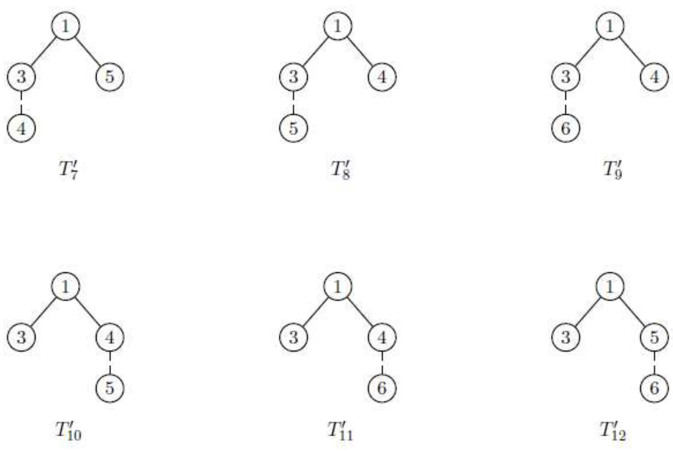

Figure 7, Subtrees T_7'-T_{12} ′ obtained from Steps 23-46 of Algorithm 2

Figure 8. Subtrees T_{13} ′ and T_{14} ′ displayed by Steps 47-50, 51

Theorem 5. The inequalities

$x_e \geq 0, \ \forall e \in E,$
$x_e \leq 1, \ \forall e \in E.$

define (trivial) facets of $P_T(G)$.

Theorem 4. Algorithm 2 constructs (m-1) p − vertex spanning subtrees that incidence vectors are affinelyindependent and satisfy$x_e \geq 0, \forall e$ with equality.

Proof. By virtue of Theorem 4, it's possible to create ($m-1$) p − vertex spanning subtrees T'_i , $i = 1, \dots, m-1$ that incidence vectors $\tau'_1, \tau'_2, \dots, \tau'_{m-1}$ satisfy an inequality of type (4), $x_e \geq 0$, with equality and are affinely independent. Moreover, consider the inequality $x_e \geq 0$, all vectors of p−vertex spanning subtrees that pass by the edge e strictly verify the inequality. This proves that an inequality of type (4) is not an equation. On the other hand, Steps 2-40 of Algorithm 1 permit to construct (m − 1) p−vertex spanning subtrees that incidence vectors $\tau'_1, \tau'_2, \dots, \tau'_{m-1}$ satisfy an inequality of type (5), $x_e \leq 1$, with equality and are affinelyindepen- dent, respectively. More, any p−vertex spanning subtree that do not pass through the edge e strictly verify the inequality $x(e) \leq 1$, showing that we do not deal with an equation. This implies that the inequalities (4) and (5) are facet defining of $P_T(G)$ and completes the proof.

Conclusion

The main contribution of this paper is the introduction of a new linear formulation for the minimum weighted spanningsubtree problem. After, we address several constructive algorithms that generate a set of subtrees spanning with affinely independent corresponding incidence vectors. Consider the polytope associated to this linear formulation and unlike the traditional approach that consists to look for the affine subspace of the subtree polytope, to determine the polytope dimension and to show the facetness of the trivial constraints (valid inequalities)defining the polytope, we resort to these algorithms.

REFERENCES

Aider, M., Aoudia, L., Baiou, M., Mahjoub, R. & Nguyen, V. H. (2019).On the star forest polytope for trees and cycles, RAIRO-Oper. Res, pp. 1763-1773.

Blum, C., Ehrgott, M. 2003. Local search algorithms for the k− cardinality tree problem. Discrete applied math-ematics 128 (2-3), pp. 511-540.

Chimani, W., Kandyba, M., Ljubic, I., Mutzel, P. 2008. Obtaining optimal k− cardianlity trees fast. Journal of experimental algorithms 14, pp. 27-36.

Chou, W., and Frank, H., 1970.Survivable communication networks and the terminal capacity matrix. IEEE Trans Circuit Theory 17, pp. 183-192.

Christofides, N., Whitlock, C.A., 1981.Network synthesis with connectivity constraints - A survey.Operationnal Research 81, J.P. Brans (Editor), North Holland, Amsterdam, pp. 705-723.

Ehrgott, M., Freitag, J., Hamacher, H. W., Maffioli, F., 1996. Heuristics for the k−cardinality tree and subgraph problems. Networks 24, pp. 11-21.

Fischetti, M., Hamacher, H. W, Jornsten, K., Maffioli, F., 1994. Weighted k− cardinality trees: Complexity and polyhedral structure. Networks 24, pp. 11-21.

Foulds, L.R. and Hamacher, H.W, 1992. A new integer programming approach to (restricted) facilities layout problem allowing flexible facility shapes. Report 1992-3, Department of Management Systems. University of Waikato.

Frank, A., 1992. Augmenting graphs to meet edge connectivity requirements.SIAM J. Discr. Math 5, pp. 25-53.

Garey, M. Y., Johnson, D. S. (1979). Computers and Intractability: A Guide to the Theory of Np-Completeness W.H. Freeman & Co Ltd.

Garg, N., 1996. A 3-approximation for the minimum tree spanning k vertices. Proc. Symposium on Foundations of Computer Science (FOCS′96). IEEE Computer Society, pp. 302-309.

Garg, N., Hochbaum, D., 1996. An O(log k) approximation algorithm for the k−minimum spanning tree problem in the plane. Algorithmica 18, pp. 111-121.

Grotschel, M. and Monma, C.L., 1990. Integer polyhedra arising from certain network design problem with connectivity constraints. SIAM Journal of Discrete mathematics, vol 3.no 4, pp. 502 - 523.

Grotschel, M. and Monma, C.L., Stoer, M., 1992. Facets for polyhedral arising in the design of communication networks with with low-connectivity constraints. SIAM Journal of Optimization, vol 2. no 3, pp. 474 - 504.

Grotschel, M. and Monma, C.L., Stoer, M., 1992.Computatonal results with a cutting plane algorithm for designing communication networks with low-connectivity constraints. Operations research, 40, pp. 309 - 330.

Grotschel, M. and Monma, C.L., Stoer, M., 1995.Design of survivable networks. Handbook in Operations Research and Management Science 7, M.O Ball C.L. Monma and G. Nemhauser (Editors), North Holland, Amsterdam, pp. 617 - 671.

Grotschel, M., Monma, C. L. &Stoer, M. (1995). Polyhedral and computational investigations for designing communication networks with high survivability requirements. Operations Research, vol 43.no 6, pp. 1012 - 1024.

Hamacher, H.W. and Joernsten, K. 1992. Optimal relinquishments. Technical Report. Norwegian School of Eco-nomics and Business Administration.15

Ibrahim, M. S., Maculan, N. & Minoux, M. (2015). Valid inequalities and lifting procedures for the shortest path problem in digraphs with negative cycles. Optimization letters, 9, pp. 345-357.

Ibrahim, M. S., Maculan, N. & Ouzia, H. (2016). An efficient cutting plane algorithm solving the minimum weighted elementary directed cycle in planar digraphs. RAIRO-Oper. Res. 50, pp. 665-675.

Maculan. N, Plateau. G and Lisser.A. (2003). Integer linear models with a polynomial number of variables and constraints for some classical combinatorial optimization problems. Pesquisa Operacional, vol. 23, (2003), pp.161-168.

Magnanti, T.L., Raghavan, S., 2005. Strong formulations for network design problems with connectivity require-ments. Networks 45 (2), pp.61-79.

Philpott, A.B. and Wormald, N.C, 1997. On the optimal extraction of ore from an open-cast mine. University of Auckland, New Zealand.

Simonetti, L., De Souza, C.C., 2011. New Branch and Bound algorithms for the k− cardinality tree problem. Electronic notes in discrete mathematics 37, pp. 27-32.

Simonetti, L., Cunha, A.S., Lucena, A., 2013. Polyhedral results and Branch and cut algorithm for the k−cardinality tree problem. Mathematical programming 142, pp. 511-538.

****