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RESEARCH ARTICLE

SOLUTION OF LINEAR HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATIONS BY NEW SUMUDU VARIATIONAL ITERATION METHOD

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ABSTRACT

In this work we propose a combined Sumudu transform (ST) and the variational iteration method (VIM) to solve linear homogeneous partial differential equations. The elegant coupling is called the Sumudutrans form variational iteration method (STVIM).A general Lagrange multiplier is used to construct a correction functional, which can be identified via variational theory. The solutions of these examples are contingent only on the initial conditions. The method is elegant and reliable.

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INTRODUCTION

Partial differential equations; linear or nonlinear, homogeneous or inhomogeneous has many applications to real life problems that arise in science, engineering and technology. There are many numerical methods for the solution of different types of differential equations such as Adomian decomposition method ,homotopy perturbation method , variational iteration method , modified variational iteration method . Results by various researchers have shown reliability, efficiency and applicability of these method. In this paper, a variational iteration method for the solution of a linear partial differential equations. The variational iteration method (VIM) was developed, in 1999, by He. This method is, now, widely used by many researchers to study linear and nonlinear problems. Motivated and inspired by Wu's thinking, and combing with the Sumudu transform (ST), we give a new modified variational iteration method (VIM),which is based on variational iteration theory and Sumudu transform (ST).The balance in this work as follows: the Sumudu transform (ST),variational iteration method (VIM), and the combination of Sumudu transform (ST) and variational iteration method (VIM) are presented in sections 2 and 3. In section 4, numerical application of the method is illustrated by two test examples to demonstrate the efficiency of the method. Section 5 includes a conclusion that briefly summarizes the results**.**Also the main result of this paper is to introduce an alternative Sumudu correction functional and express the integral as a convolution.

Sumudu Transfrom (ST): The Sumudu transform is an integral transform, which was first introduced by watugala to solve differential equations and control engineering problems.

Definition: The Sumudutrans form of a function $f(t)$is defined over the set of functions,

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$$A = \left\{ f(t) \backslash \exists M, \tau_1, \tau_2 > 0, |f(x)| < Me^{\frac{|t|}{\tau_j}}, if\ t\epsilon(-1)^j \times [0, \infty) \right\}$$

by,

$$G(u) = S\{f(t)\} = \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}} f(t)dt,\ u \in (-\tau_1, \tau_2) \tag{1}$$

The inverse of Sumudu transform of function $G(u)$ is denoted by symbol $S^{-1}[G(u)] = f(t)$ and is defined with Bromwich contour integral by

$$S^{-1}[G(u)] = f(t) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{\gamma-iT}^{\gamma+iT} e^{st}\ G(u)\, du$$

Proposition 1 It deals with the effect of the differentiation of the function $f(x,t)$, k times on the Sumudu transform $G(x,u)$ if $S\{f(x,t)\} = G(x,u)$ then

- $S\left(\frac{\partial f(x,t)}{\partial x}\right) = \frac{d}{dx} G(x,u)$
- $S\left(\frac{\partial f(x,t)}{\partial t}\right) = \frac{1}{u}(G(x,u) - f(x,0))$
- $S\left(\frac{\partial f(x,t)}{\partial x \partial t}\right) = \frac{1}{u}\frac{d}{dx}(G(x,u) - f(x,0))$
- $S\left(\frac{\partial^2 f(x,t)}{\partial x^2}\right) = \frac{d^2}{dx^2} G(x,u)$
- $S\left(\frac{\partial^2 f(x,t)}{\partial t^2}\right) = \frac{1}{u^2} G(x,u) - \frac{1}{u^2} f(x,0) - \frac{1}{u}\frac{\partial f(x,0)}{\partial t}$

We can easily extend this result to the nth partial derivative by mathematical induction

(vi) $$S\left(\frac{\partial^n f(x,t)}{\partial t^n}\right) = u^{-n}\left(G(x,u) - \sum_{k=0}^{n-1} u^k (x,u) f^k(x,0)\right)$$

Sumudu Transform Variational Iteration Method (STVIM)

Consider the following general differential equations

$$L[w(x,t)] + N[w(x,t)] = f(x,t),$$

Where L a linear partial differential operator given by $\frac{\partial^2}{\partial t^2}$, N is a nonlinearoperator, and $f(x,t)$ is a known analytical function. We can construct a correction functional according to the variational iteration method (VIM) for Eq. (2)as follows:

$$w_{n+1}(x,t) = w_n(x,t) + \int_0^t \lambda(x,\varepsilon)(Lw_n(x,\varepsilon) + N\,\widetilde{w_n}(x,\varepsilon) - f(x,\varepsilon))\, d\varepsilon,\ \ n \geq 0 \tag{3}$$

Where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, $w_n(x,\varepsilon)$ is the nth approximate solution and $\widetilde{w_n}(x,\varepsilon)$ is a restricted variation which means $\delta\widetilde{w_n}(x,\varepsilon) = 0$.

In a wide range of problems that appear in the literature, the general form of Lagrange multiplier is found to be of the form:

$\lambda = \bar{\lambda}(x, t-\varepsilon)$.

In this section, we will make the assumption that λ is expressed in this latter way. In such a case, the integration is basically the convolution; hence Sumudu transform (**ST**) is appropriate to use. Applying Sumudu transform (**ST**) on both sides of (3) the correction functional will be constructed in the following manner:

$$S(w_{n+1}(x,t)) = S(w_n(x,t)) + S\left(\int_0^t \bar{\lambda}(x,\varepsilon)(Lw_n(x,\varepsilon) + N\,\widetilde{w_n}(x,\varepsilon) - f(x,\varepsilon))\, d\varepsilon\right), n \geq 0 \tag{4}$$

Therefore

$$S(w_{n+1}(x,t)) = S(w_n(x,t)) + S\left(\bar{\lambda}(x,t) * (Lw_n(x,t) + N\,\widetilde{w_n}(x,t) - f(x,t))\right) \tag{5}$$

$$\mathcal{S}\big(w_{n+1}(x,t)\big) = \mathcal{S}\big(w_n(x,t)\big) + u\mathcal{S}\left(\bar{\lambda}(x,t)\right) * \mathcal{S}\big(Lw_n(x,t) + N\,\widetilde{w_n}(x,t) - f(x,t)\big) \tag{6}$$

To find the optimal value of $\bar{\lambda}(x, t-\varepsilon)$ we first take the variation with respect to $w_n(x,t)$ .Thus

$$\frac{\delta}{\delta w_n}\mathcal{S}\big(w_{n+1}(x,t)\big) = \frac{\delta}{\delta w_n}\mathcal{S}\big(w_n(x,t)\big) + u\frac{\delta}{\delta w_n}\mathcal{S}\left(\bar{\lambda}(x,t)\right) * \mathcal{S}\big(Lw_n(x,t) + N\,\widetilde{w_n}(x,t) - f(x,t)\big) \tag{7}$$

And hence upon applying the variation this simplifies to

$$\mathcal{S}\big(\delta w_{n+1}(x,t)\big) = \mathcal{S}\big(\delta w_n(x,t)\big) + u\mathcal{S}\left(\bar{\lambda}(x,t)\right) * \mathcal{S}\big(\delta w_n(x,t)\big) \tag{8}$$

In this paper, we assume that L is a linear partial differential operator given by $\frac{\partial^2}{\partial t^2}$, then Eq.(8) becomes:

$$\mathcal{S}\big(\delta w_{n+1}(x,t)\big) = \mathcal{S}\big(\delta w_n(x,t)\big) + u\mathcal{S}\left(\bar{\lambda}(x,t)\right)\left(\tfrac{1}{u^2}\right)\mathcal{S}\big(\delta w_n(x,t)\big), \tag{9}$$

The extremum condition of w_{n+1} requires that $\delta w_{n+1} = 0.$ This means that the right -hand side Eq.(9) should be set to zero. Hence, we have the stationary condition

$$\mathcal{S}\left(\bar{\lambda}(x,t)\right) = -u \tag{10}$$

Taking the Sumudu inverse of the last equation gives the optimal value of $\bar{\lambda} = -t.$ For this value of $\bar{\lambda}$, we have the following formulation:

$$\mathcal{S}\big(w_{n+1}(x,t)\big) = \mathcal{S}\big(w_n(x,t)\big) - \mathcal{S}\left(\int_0^t (t-\varepsilon)\big(Lw_n(x,\varepsilon) + N\,\widetilde{w_n}(x,\varepsilon) - f(x,\varepsilon)\big)\,d\varepsilon\right), n \geq 0$$

Applications: In this section, we apply the Sumudu variational iteration method for solving linearhomogeneouspartial differential equations.

Example1. Consider the following linearhomogeneouspartialdifferential equation

$$w_{tt}(x,t) - w_{xx}(x,t) + w(x,t) = 0, \tag{11}$$

$$w(x,0) = 0, \quad \frac{\partial w(x,0)}{\partial t} = x.$$

The exact solution for the givendifferential equation is $w = x\,sint$

For this case the Sumudu variational iteration correction functional will be constructed in the following manner:

$$w_{n+1}(x,t) = w_n(x,t) + \int_0^t \bar{\lambda}(x, t-\varepsilon)\,[(w_n)_{tt}(x,\varepsilon) - (w_n)_{xx}(x,\varepsilon) + w_n(x,\varepsilon)]d\varepsilon, \; n \geq 0 \tag{12}$$

Next, by applying Sumudu transform, we have:

$$\mathcal{S}\big(w_{n+1}(x,t)\big) = \mathcal{S}\big(w_n(x,t)\big) + \mathcal{S}\left(\int_0^t \bar{\lambda}(x,t-\varepsilon)\,[(w_n)_{tt}(x,\varepsilon) - (w_n)_{xx}(x,\varepsilon) + w_n(x,\varepsilon)]d\varepsilon\right). \tag{13}$$

or equivalent, by applying the convolution property, we obtain:

$$\mathcal{S}\big(w_{n+1}(x,t)\big) = \mathcal{S}\big(w_n(x,t)\big) + u\mathcal{S}(\bar{\lambda}) * \mathcal{S}\big((w_n)_{tt}(x,t) - (w_n)_{xx}(x,t) + w_n(x,t)\big), \tag{14}$$

$$\mathcal{S}\big(w_{n+1}(x)\big) = \mathcal{S}\big(w_n(x,t)\big) + u\mathcal{S}(\bar{\lambda})\left[\frac{1}{u^2}\mathcal{S}\big(w_n(x)\big) - \frac{w_n(x,0)}{u^2} - \frac{1}{u}\frac{\partial w_n(x,0)}{\partial t} - \mathcal{S}(w_n)_{xx}(x,t) + \mathcal{S}\big(w_n(x,t)\big)\right]. \tag{15}$$

Applying the variation on the Eq. (15), we get

$$\frac{\delta}{\delta w_n} S\big(w_{n+1}(x,t)\big) = \frac{\delta}{\delta w_n} S\big(w_n(x,t)\big) + \frac{\delta}{\delta w_n} u\, S\left(\bar{\lambda}(x)\right) * \left[\frac{1}{u^2} S\big(w_n(x)\big) - \frac{w_n(x,0)}{u^2} - \frac{1}{u}\frac{\partial w_n(x,0)}{\partial t} - S(w_n)_{xx}(x,t) + S\big(w_n(x,t)\big)\right]. \tag{16}$$

By simplifying Eq. (16), we get

$$S\big(\delta w_{n+1}(x,t)\big) = S\big(\delta w_n(x,t)\big) + uS\left(\bar{\lambda}(x)\right)\left(\tfrac{1}{u^2}+1\right) S\big(\delta w_n(x)\big) \tag{17}$$

$$S\big(\delta w_{n+1}(x,t)\big) = S\big(\delta w_n(x,t)\big)\left[1 + \left(\tfrac{1+u^2}{u}\right) S\left(\bar{\lambda}(x)\right)\right] \tag{18}$$

The extremum condition of $w_{n+1}(x,t)$ requires that $\delta w_{n+1}(x,t) = 0$, then

$$S\big(\delta w_n(x,t)\big)\left[1 + \left(\frac{1+u^2}{u}\right) S\left(\bar{\lambda}(x)\right)\right] = 0$$

$$S\left(\bar{\lambda}(x)\right) = -\frac{u}{1+u^2} \tag{19}$$

Applying the inverse Sumudu transform, we get:

$$\bar{\lambda}(x) = -\sin t \tag{20}$$

Substituting Eq.(20) into Eq.(13), we get

$$S\big(w_{n+1}(x,t)\big) = S\big(w_n(x,t)\big) - S\left(\int_0^t \sin(t-\varepsilon)\,[(w_n)_{tt}(x,\varepsilon) - (w_n)_{xx}(x,\varepsilon) + w_n(x,\varepsilon)]\,d\varepsilon\right), \tag{21}$$

or equivalent, by applying the convolution property, we obtain:

$$S\big(w_{n+1}(x,t)\big) = S\big(w_n(x,t)\big) - uS(\sin t) * S((w_n)_{tt}(x,t) - (w_n)_{xx}(x,t) + w_n(x,t)), \tag{22}$$

Suppose that $w_0(x,t) = w(x,0) + t\frac{\partial w(x,0)}{\partial t} = xt,$ then from Eq.(22)we have:

$$S\big(w_1(x,t)\big) = S(xt) - uS(\sin t) * S(xt),$$

$$S\big(w_1(x,t)\big) = xu - u\left(\frac{u}{1+u^2}\right)(xu),$$

or

$$S\big(w_1(x,t)\big) = \frac{xu}{1+u^2} \tag{23}$$

Applying the inverse of Sumudu transform, we have

$$w_1(x,t) = x\sin t \tag{24}$$

Substituting Eq.(24)into Eq.(22), we get

$$S\big(w_2(x,t)\big) = S(x\sin t) - uS(\sin t) * S(0),$$

$$S\big(w_2(x,t)\big) = \frac{xu}{1+u^2},$$

Applying the inverse of Sumudu transform, we have

$$w_2(x,t) = x\sin t \tag{25}$$

Therefore, the exact solution of given linearhomogeneouspartial differential equation is $w(x,t) = x\sin t$

Example2. Consider the following linear homogeneouspartial differential equation

$$w_{tt}(x,t) - w_{xx}(x,t) = 0, -\infty < x < \infty, t > 0 \tag{26}$$

$w(x,0) = 0, \quad \frac{\partial w(x,0)}{\partial t} = x^2.$

The exact solution for the given differential equation is $w = x^2 t + \frac{t^3}{3}$

For this case the Sumudu variational iteration correction functional will be constructed in the following manner:

$$w_{n+1}(x,t) = w_n(x,t) + \int_0^t \bar{\lambda}(x,t-\varepsilon)\,[(w_n)_{tt}(x,\varepsilon) - (w_n)_{xx}(x,\varepsilon)]d\varepsilon,\ n \geq 0 \quad (27)$$

Next, by applying Sumudu transform, we have:

$$\mathcal{S}\big(w_{n+1}(x,t)\big) = \mathcal{S}\big(w_n(x,t)\big) + \mathcal{S}\left(\int_0^t \bar{\lambda}(x,t-\varepsilon)\,[(w_n)_{tt}(x,\varepsilon) - (w_n)_{xx}(x,\varepsilon)]d\varepsilon\right), \quad (28)$$

or equivalent, by applying the convolution property, we obtain:

$$\mathcal{S}\big(w_{n+1}(x,t)\big) = \mathcal{S}\big(w_n(x,t)\big) + u\mathcal{S}(\bar{\lambda}) * \mathcal{S}\big((w_n)_{tt}(x,t) - (w_n)_{xx}(x,t)\big), \quad (29)$$

$$\mathcal{S}(w_{n+1}(x)) = \mathcal{S}(w_n(x,t)) + u\mathcal{S}(\bar{\lambda})\left[\frac{1}{u^2}\mathcal{S}(w_n(x)) - \frac{w_n(x,0)}{u^2} - \frac{1}{u}\frac{\partial w_n(x,0)}{\partial t} - \mathcal{S}(w_n)_{xx}(x,t)\right]. \quad (30)$$

Applying the variation on the Eq. (30), we get

$$\frac{\delta}{\delta w_n}\mathcal{S}(w_{n+1}(x,t)) = \frac{\delta}{\delta w_n}\mathcal{S}(w_n(x,t)) + \frac{\delta}{\delta w_n}u\,\mathcal{S}(\bar{\lambda}(x)) * \left[\frac{1}{u^2}\mathcal{S}(w_n(x)) - \frac{w_n(x,0)}{u^2} - \frac{1}{u}\frac{\partial w_n(x,0)}{\partial t} - \mathcal{S}(w_n)_{xx}(x,t)\right]. \quad (31)$$

By simplifying Eq. (31), we get

$$\mathcal{S}\big(\delta w_{n+1}(x,t)\big) = \mathcal{S}\big(\delta w_n(x,t)\big) + u\mathcal{S}\left(\bar{\lambda}(x)\right)\frac{1}{u^2}\,\mathcal{S}\big(\delta w_n(x)\big) \quad (32)$$

$$\mathcal{S}\big(\delta w_{n+1}(x,t)\big) = \mathcal{S}\big(\delta w_n(x,t)\big)\left[1 + \frac{1}{u}\,\mathcal{S}\left(\bar{\lambda}(x)\right)\right] \quad (33)$$

The extremum condition of $w_{n+1}(x,t)$ requires that $\delta w_{n+1}(x,t) = 0$, then

$$\mathcal{S}\big(\delta w_n(x,t)\big)\left[1 + \frac{1}{u}\,\mathcal{S}\left(\bar{\lambda}(x)\right)\right] = 0$$

$$\mathcal{S}\left(\bar{\lambda}(x)\right) = -u \quad (34)$$

Applying the inverse Sumudu transform, we get:

$$\bar{\lambda}(x) = -t \quad (35)$$

Substituting Eq.(35)into Eq.(28), we get

$$\mathcal{S}\big(w_{n+1}(x,t)\big) = \mathcal{S}\big(w_n(x,t)\big) - \mathcal{S}\left(\int_0^t t\,[(w_n)_{tt}(x,\varepsilon) - (w_n)_{xx}(x,\varepsilon)]d\varepsilon\right), \quad (36)$$

or equivalent, by applying the convolution property, we obtain:

$$\mathcal{S}\big(w_{n+1}(x,t)\big) = \mathcal{S}\big(w_n(x,t)\big) - u\mathcal{S}(t) * \mathcal{S}\big((w_n)_{tt}(x,t) - (w_n)_{xx}(x,t)\big), \quad (37)$$

Suppose that $w_0(x,t) = w(x,0) + t\frac{\partial w(x,0)}{\partial t} = x^2 t,$ then from Eq.(37)we have:

$$\mathcal{S}\big(w_1(x,t)\big) = \mathcal{S}(x^2 t) - u\mathcal{S}(t) * \mathcal{S}(-2t),$$

$$\mathcal{S}\big(w_1(x,t)\big) = x^2 u - u(u)(-2u),$$

or

$$\mathcal{S}\big(w_1(x,t)\big) = x^2 u + 2u^3 \quad (38)$$

Applying the inverse of Sumudu transform, we have

$w_1(x,t) = x^2 t + \frac{t^3}{3}$ (39)

Substituting Eq.(39)into Eq.(37), we get

$$S(w_2(x,t)) = S\left(x^2 t + \frac{t^3}{3}\right) - uS(t) * S(0),$$

$$S(w_2(x,t)) = x^2 u + 2u^3$$

Applying the inverse of Sumudu transform, we have

$$w_2(x,t) = x^2 t + \frac{t^3}{3} \quad (40)$$

Therefore, the exact solution of given linearhomogeneouspartial differential equation is

$$w(x,t) = x^2 t + \frac{t^3}{3}$$

Conclusion

In this paper, Sumudu transform variational iteration methodhas been efficiently applied for solving linear partial differential equationsto give rapid convergent successive approximations without any linearization and successfully implemented by using initial conditions and convolution integral. It is also clear and remarkable that approximate solutions are in good agreement with analytical solution.

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