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RESEARCH ARTICLE

USE OF LAPLACE TRANSFORM IN DIFFERENTIAL EQUATIONS

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ABSTRACT

The concept of Laplace transform plays an important role in various fields of science, engineering and technology such as control engineering, communication, signal analysis and design, system analysis, solving differential equations, system of modeling, etc. In this paper we have to discuss the method of solution of differential equations using Laplace transform.

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INTRODUCTION

In mathematics an integral transform plays an important role in the conversion of a function from one function into another function. The Laplace transform method can be used for finding solution of system of ordinary differential equations, partial differential equations and integral equations. It can be used to convert many common functions such as exponential functions, sinusoidal functions and damped sinusoidal functions into algebraic functions of a complex variable s. Operations such as differentiation and integration can be replaced by algebraic operations in the complex plane. Thus the linear differential equation can be transformed into algebraic functions of a complex variable s. For finding the solution of differential equation the system described by ODE the solution is difficult. Hence we consider system described by Transfer Function. The method is very easy to explain. Apply the Laplace transform on both sides of the given differential equation. This will transformation the differential equation into algebraic equation. If we solve this equation for roots then taking the inverse Laplace transform on both sides. The result is the solution of given differential equation.

Preliminaries

Definition: Laplace Transform: The Laplace transform of a function f(y) defined for all real numbers $y \geq 0$, is the function F(s), which is a unilateral transform defined by

$L[f(y)] = F(s) = \int_0^{\infty} f(y)e^{-s}\, dy$ where s is real or complex number frequency parameter

Properties of Laplace Transform:

1) **Linearity property:** If C_1 and C_2 are any constants, $f_1(y)$ and $f_2(y)$ are functions with Laplace transform $F_1(s)$ and $F_2(s)$ respectively

i.e. $L[f_1(y)] = F_1(s)$ and $L[f_2(y)] = F_2(s)$, then

$$L\{C_1 f_1(y) + C_2 f_2(y)\} = C_1 L\{f_1(y)\} + C_2 L\{f_2(y)\} = C_1 F_1(s) + C_2 F_2(s)$$

2) **First translation property:** If $L[f(y)] = F(s)$ then

$L\,[e^{a}\ f(y))] = F(s-a),\quad s-a>a$

3) **Second translation property:** If $L[f(y)] = F(s)$ and $g(y) = \begin{cases} F(y-a), & y < a \\ 0 & , y > a \end{cases}$ then

$L[g(y))] = e^{-a}\ y(s)$

4) **Change of scale property:** If $L[f(y)] = F(s)$ then $L[f(ay)] = \frac{1}{a} F\left(\frac{s}{a}\right)$

5) **Laplace transform of Derivatives:** If $L[f(y)] = F(s)$ then

i) $L[f'(y)] = s\,L[f(y)] - f(0)$

In general ii) $L[f^{(n)}(y)] = s^n L[f(y)] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$

6) **Multiplication of y:** If L[f(y)] = F(s) then

i) $L[yf(y)] = -F'(s)$
ii) $L[yf'(y)] = -sF'(s) - F(s)$
iii) $L[yf''(y)] = -s^2F'(s) - 2sF(0) - f(0)$

7) **Multiplication of y^n:** If L[f(y)] = F(s) then $L[y^n f(y)] = (-1)^n \frac{d^n}{ds^n}F(s) = (-1)^n F^{(n)}(s)$

8) **Laplace transform of Integrals:** If L[f(y)] = F(s) then

$L[\int_0^\infty f(u)du] = \frac{F(s)}{s}$

9) **Division by y:** If L[f(y)] = F(s) then $L[\frac{f(y)}{s}] = \int_0^\infty F(u)du$

Laplace Transform of some standard Functions:

Table 1.

f(y)	F(s) = L{f(y)}
1	$\frac{1}{s}$
y	$\frac{1}{s^2}$
y^n	$\frac{n!}{s^{n+1}}$, (n = 0,1,2,⋯)
e^a	$\frac{1}{s-a}$
$\frac{1}{a}(e^a - 1)$	$\frac{1}{s(s-a)}$
sin(ay)	$\frac{a}{s^2+a^2}$
cos(ay)	$\frac{s}{s^2+a^2}$
sinh(ay)	$\frac{a}{s^2-a^2}$
cosh(ay)	$\frac{s}{s^2-a^2}$
e^b sin(ay)	$\frac{a}{(s-b)^2+a^2}$
e^b cos(ay)	$\frac{s-b}{(s-b)^2+a^2}$
ysin(ay)	$\frac{2as}{(s^2+a^2)^2}$
ycos(ay)	$\frac{s^2-a^2}{(s^2+a^2)^2}$

Definition: Inverse Laplace Transform: If F(s) is the Laplace transform of f(y) then the inverse Laplace transform of F(s) is given by f(y) and we write,

If L[f(y)] = F(s) then $L^{-1}[F(s)] = f(y)$.

Properties of Inverse Laplace Transform:

1) **Inverse Laplace transform Linearity property:** If C_1 and C_2 are any constants, $f_1(y)$ and $f_2(y)$ are functions with Laplace transform $F_1(s)$ and $F_2(s)$ respectively, then

$$L^{-1}\{C_1F_1(s) + C_2F_2(s)\} = C_1L^{-1}\{F_1(s)\} + C_2L^{-1}\{F_2(s)\} = C_1f_1(y) + C_2f_2(y)$$

2) **Inverse Laplace transform First translation property:** If $L^{-1}\{F(s)\} = f(y)$ then

$L^{-1}\{F(s-a)\} = e^a f(y)$, s-a>a

3) **Inverse Laplace transform Second translation property:** If $L^{-1}\{F(s)\} = f(y)$ then $L^{-1}\{e^{-a}F(s)\} = \begin{cases} f(y-a), & y > a \\ 0 & , y < a \end{cases}$

4) **Inverse Laplace transform change of scale:** If $L^{-1}\{F(s)\} = f(y)$ then

$L^{-1}\{F(ks)\} = \frac{1}{k} f\left(\frac{y}{k}\right)$

5) **Inverse Laplace transform of Derivatives:** If $L^{-1}\{F(s)\} = f(y)$ then

$L^{-1}\{F^n(s)\} = (-1)^n y^n f(y)$

6) **Inverse Laplace transform of Integrals:** If $L^{-1}\{F(s)\} = f(y)$ then

$L^{-1}\left\{\int_0^\infty F(s)ds\right\} = \frac{f(y)}{y}$

7) **Inverse Laplace transform Multiplication of s :** If $L^{-1}\{F(s)\} = f(y)$ and

i) If f(0) = 0 then $L^{-1}\{sF(s)\} = f'(y)$
ii) If f(0) ≠ 0 then $L^{-1}\{sF(s) - f(0)\} = f'(y)$

or $L^{-1}\{sF(s)\} = f'(y) + f(0)\,\delta(y)$ where δ(y) is the unit impulse function.

8) **Inverse Laplace transform Division by s:** If $L^{-1}\{F(s)\} = f(y)$

then $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^\infty f(u)du$

9) **Inverse Laplace transform of Multiplication of two functions (Convolution of F_1 and F_2):** If $f_1(y)$ and $f_2(y)$ are functions with Laplace transform $F_1(s)$ and $F_2(s)$ respectively

i.e. $F_1(s) = L\{f_1(y)\}$ and $F_2(s) = L\{f_2(y)\}$, then $L\{f_1(y) * f_2(y)\} = F_1(s) \cdot F_2(s)$ or $f_1(y) * f_2(y) = L^{-1}\{F_1(s) \cdot F_2(s)\}$

MAIN RESULTS

Solving Ordinary Differential Equations

Problem: $Y'' + aY' + bY = g(t)$ Subject to the condition Y(0) = A, Y′(0) = B where a, b, A,B are constants.

Solution:

1) Write the differential equation
2) Take Laplace transform on both sides of the given differential equation. We use the properties of derivative as necessary.
3) Put initial conditions into the resulting equation
4) Obtain an equation L[f(y)] = F(s) is an algebraic expression in the variable s and solve this equation.
5) Apply the inverse Laplace transform we get the solution.

Example: Solve the differential equation $ty'' + y = 0$ given that y (0) = 0

Solution: Consider the differential equation $ty'' + y = 0$ --------- (1)

Given that y (0) = 0

Taking the Laplace transform on both sides of equation (1) we get,
$L\{ty'' + y\} = 0 \Rightarrow L\{ty''\} + L\{y\} = 0$

$\Rightarrow -\frac{d}{d} L\{y''\} + L\{y\} = 0$ $L[t^n y(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$

$\Rightarrow \frac{d}{d}\{s^2 L(y) - sy(0) - y'\} = L\{y\}$ Laplace transform of derivatives

$\Rightarrow \frac{d}{d}\{s^2 z - 0 - k\} = z$ $y(0) = 0$, $y'(0) = k$ and $L(y) = z$

$\Rightarrow s^2 \frac{d}{d} + 2s\, z - 0 = z$

$\Rightarrow s^2 \frac{d}{d} = (1 - 2s)z$

$\Rightarrow \frac{d}{z} = \{\frac{1}{s^2} - \frac{2s}{s^2}\}\, ds$

On integrating we get,

$\log z = \frac{-1}{s} - 2\log s + \log c \Rightarrow \log\left(\frac{z s^2}{c}\right) = \frac{-1}{s}$

$\Rightarrow \frac{z s^2}{c} = e^{\frac{-1}{s}}$

$\Rightarrow z = \frac{c}{s^2} e^{\frac{-1}{s}}$

$\Rightarrow z = \frac{c}{s^2} \sum_{n=0}^{\infty} \frac{(\frac{-1}{s})^n}{n!}$ $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\Rightarrow z = \frac{c}{s^2} \sum_{n-0}^{\infty} \frac{(-1)^n}{n!\, s^n}$

$\Rightarrow L(y) = c \sum_{n-0}^{\infty} \frac{(-1)^n}{n!\, s^{n+2}}$

$\Rightarrow y = L^{-1}\{c \sum_{n-0}^{\infty} \frac{(-1)^n}{n!\, s^{n+2}}\}$

$\Rightarrow y = c \sum_{n-0}^{\infty} \frac{(-1)^n t^{n+1}}{n!(n+1)!}$

Conclusion

In different areas of mathematics, physics and engineering the Laplace transform is very powerful tool. The Laplace transforms are used to reduce a differential equation to a simple equation in s space and a system of differential equations to a system of linear equations. The system described by transform function the solution is easier as compared to system described by ordinary differential equations. In Laplace transform the solution of a differential equation is directly obtained without first determined general solution. No homogeneous differential equation can be solved without first solving the corresponding homogeneous equation.

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