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RESEARCH ARTICLE

THEORITICAL ANALYSIS OF THE GENERALIZED ODD LINDLEY-GOMPERTZ DISTRIBUTION

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ABSTRACT

In this article, we proposed another extension of the Gompertz distribution called the "Generalized odd Lindley-Gompertz distribution''. The probability density function (pdf) and the cumulative density function (cdf) of the new distribution were defined using the idea of Odd-Lindley G family proposed by Gomes-Silva et, al (2017). Analytical expressions for some mathematical quantities comprising moments, moment generating function, characteristics function and order statistics were presented. Some properties of the new distribution have been derived and discussed and the method of maximum likelihood is used to estimate the parameter of the proposed distribution. The performance of the proposed model has been evaluated by using real life dataset. Finally, graphical illustration and analysis are presented with recommendation for application.

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INTRODUCTION

The art of proposing generalized classes of distributions has attracted theoretical and Applied statisticians due to their flexible properties. Most of the generalizations are Developed for one or more of the following reasons: a physical or statistical theoretical Argument to explain the mechanism of the generated data, an appropriate model that has Previously been used successfully, and a model whose empirical fit is good to the data. The probability density function and cumulative distribution function of the generalized odd lindley-Gompertz distribution can be defined as;

$$G(x)=1-\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x} \tag{1.3}$$

And

$$g(x)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x} \tag{1.4}$$

respectively.

For $x>0,\theta>0,$

where θ is the scale parameter of the Lindley distribution. The Gompertz distribution (*GD*) is both skewed to the right and to the left.

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It is an extension of the exponential distribution (*ED*) and is commonly used in many applied problems, particularly in lifetime data analysis (Johnson *et al*., 1995). The *GD* is applied in the analysis of survival, in some sciences such as gerontology (Brown and Forbes 1974), computer (Ohishi *et al.,* 2009), biology (Economos 1982), and marketing science (Bemmaor and Glady 2012). The hazard rate function of *GD* is an increasing function and often applied to describe the distribution of adult life spans by actuaries and demographers (Willemse & Koppelaar 2000). The Gompertz distribution with parameters α *and* β has the cumulative distribution function (*cdf*) and probability density function (*pdf*) given by:

$$G(x)=1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)} \tag{1.5}$$

and

$$g(x)=\alpha e^{\beta x} e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)} \tag{1.6}$$

respectively.

For $x \geq 0, \alpha > 0, \beta > 0$ where α *and* β are the model parameters respectively.

2. **The Generalized Odd Lindley-Gompertz Distribution.** *(GOLGD)*

The cumulative distribution function (*cdf*) of the Odd Lindley-G family of distributions according to Gomes-Silva *et al*. (2017) is defined as,

$$F_{OL-G}(x;\theta,\xi)=\int_{-\infty}^{\frac{G(x;\xi)}{G'(x;\xi)}} \frac{\theta^2}{\theta+1}(1+t)e^{-\theta t}dt$$

For

θ= is the additional scale parameter of the Lindley distribution
α= is the scale parameter of the Linley distribution
β=is the shape parameter of the Odd-Lindley distribution
t= is a real number.
Where $G(x;\xi)$ is the cdf of any continuous distribution which depends on the parameter vector ξ, $G'(x;\xi)=1-G(x;\xi)$ and $\Theta>0$ is the scale parameter.

Using integration by substitution in the equation above and evaluating the integrand in equation yields

$$F_{OL-G}(x;\theta,\xi)=1-\frac{\theta+G'(x;\xi)}{(1+\theta)G'(x;\xi)}\exp\left\{-\theta\left[\frac{G(x;\xi)}{G'(x;\xi)}\right]\right\},-\infty<x<\infty,\theta>0 \tag{3.1}$$

Therefore, equation (3.1) is the cumulative distribution function (*cdf*) of the Odd Lindley-G family of distributions proposed by Gomes-Silva *et al*. (2017) and the corresponding *pdf* of the Odd Lindley-G family can be obtained from equation (3.1) by taking the derivative of the *cdf*

with respect to *x* and is obtained as:

$$f_{OL-G}(x;\theta,\xi)=\frac{\theta^2 g(x;\xi)}{(1+\theta)(G'(x;\xi))^3}\exp\left\{-\theta\left[\frac{G(x;\xi)}{G'(x;\xi)}\right]\right\} \tag{3.2}$$

where $g(x;\xi)$ and $G(x;\xi)$ are the *pdf* and the *cdf* of any continuous distribution respectively which depends on the parameter vector ξ and $\theta>0$ is the scale parameter. The major benefit of (3.2) is to offer more flexibility to extremes of the *pdfs* and therefore it becomes suitable for analyzing data with high degree of asymmetry. The Gompertz distribution with parameters α>0 and β>0 has the cumulative distribution function (*cdf*) and probability density function (*pdf*) given by:

$$G(x)=1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)} \tag{3.3}$$

And

$$g(x)=\alpha e^{\beta x} e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)} \tag{3.4}$$

Respectively.

For where α *and* β are the model parameters respectively.

Using equation (3.3) and (3.4) in (3.1) and (3.2) and simplifying, we obtain the *cdf* and *pdf* of the Odd Lindley-Gompertz distribution (*OLnGD*) as follows:

$$F(x)=1-\frac{\theta+\left(1-\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right)}{(1+\theta)\left(1-\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right)}\exp\left\{-\theta\left[\frac{\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)}{\left(1-\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right)}\right]\right\}$$

$$F(x)=1-\frac{\theta+e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}{(1+\theta)e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}\exp\left\{-\theta\left[\frac{1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}{e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}\right]\right\}$$

$$F(x)=1-\frac{\theta+e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}{(1+\theta)e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}\exp\left\{-\theta\left[e^{\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}-1\right]\right\} \tag{3.5}$$

And

$$f(x)=\frac{\theta^2 g(x;\xi)}{(1+\theta)(G'(x;\xi))^3}\exp\left\{-\theta\left[\frac{G(x;\xi)}{G'(x;\xi)}\right]\right\}$$

$$f(x)=\frac{\theta^2\left(\alpha e^{\beta x} e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)}{(1+\theta)\left(1-\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right)^3}\exp\left\{-\theta\left[\frac{\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)}{\left(1-\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right)}\right]\right\}$$

$$f(x)=\frac{\alpha\theta^2 e^{\beta x} e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}{(1+\theta)\left(e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^3}\exp\left\{-\theta\left[\frac{\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)}{\left(e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)}\right]\right\}$$

$$f(x)=\frac{\alpha\theta^2 e^{\beta x} e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}{(1+\theta)e^{-3\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}\exp\left\{-\theta\left[e^{\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}-1\right]\right\}$$

$$f(x)=\frac{\alpha\theta^2 e^{\beta x} e^{2\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}{(1+\theta)}\exp\left\{-\theta\left[e^{\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}-1\right]\right\} \tag{3.6}$$

respectively. Hence equation (3.5) and (3.6) are the cdf and pdf of the Odd Lindley-Gompertz distribution.

The following is a graphical representation of the *pdf* and *cdf* of the Odd Lindley-Gompertz distribution. Given some values for the parameters α, β and Θ, we provide some possible shapes for the *pdf* and the *cdf* of the *OLnGD* as shown in figure 3.1 and 3.2 below:

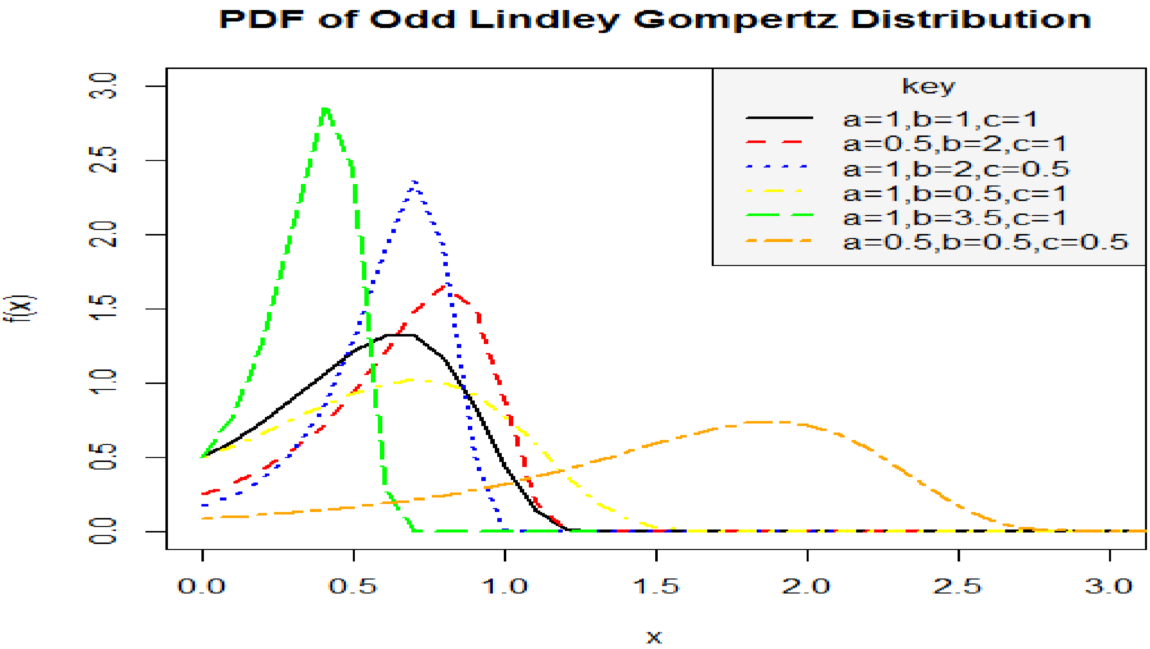

Fig. 3.1. ***PDF*** **of the** ***OL n GD*** **for different values of** $a=\alpha, b=\beta \,\&\, c=\theta$ **as shown on the key in the plot above**

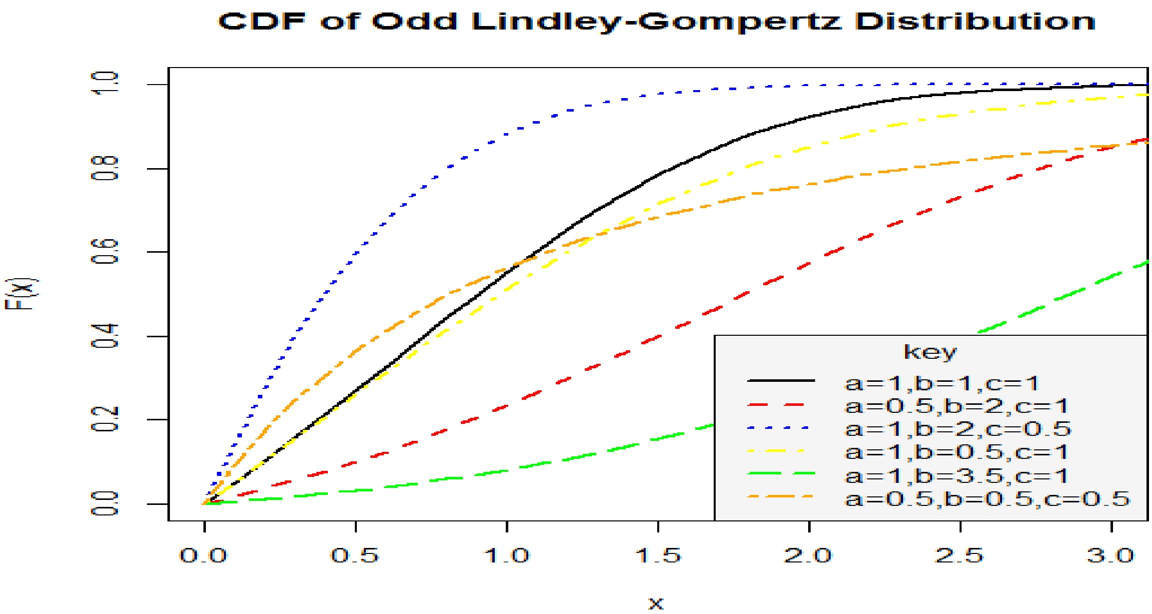

Fig. 3.2. ***CDF*** **of the** ***OL n GD*** **for different values of** $a=\alpha, b=\beta \,\&\, c=\theta$ **as shown on the key in the plot above**

From the above *cdf* plot, the *cdf* increases when *X* increases, and approaches 1 when *X* becomes large, as expected.

3.2 Properties of the Propose Distribution

3.2.1 Moments

Moments of a random variable are very important in distribution theory because some moments are used to study some of the most important features and characteristics of a random variable such as mean, variance, skewness and kurtosis.
Let X denote a continuous random variable, the n^{th} moment of X is given by;

$$\mu_n^{'} = E\left[X^n\right] = \int_0^\infty x^n f(x)dx \tag{3.7}$$

Taking *f(x)* to be the *pdf* of the Odd Lindley-Gompertz distribution as given in equation (3.6) and substituting in equation (3.7), we get the following results.

$$\mu_n^{'} = E\left[X^n\right] = \int_0^\infty x^n f(x)dx$$

$$f(x) = \frac{\theta^2\left(\alpha e^{\beta x} e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)}{(1+\theta)\left(1-\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right)^3}\exp\left\{-\theta\left[\frac{\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)}{\left(1-\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right)}\right]\right\} \tag{3.8}$$

Expansion and simplification of the pdf

$$f(x) = \frac{\theta^2\left(\alpha e^{\beta x} e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)}{(1+\theta)\left(1-\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right)^3}\exp\left\{-\theta\left[\frac{\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)}{\left(1-\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right)}\right]\right\}$$

By expanding the exponential term in (3.8) using power series, we obtain:

$$\exp\left\{-\theta\left[\frac{\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)}{\left(1-\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right)}\right]\right\} = \sum_{k=0}^{\infty}\frac{(-1)^k(\theta)^k}{k!}\left(\frac{\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)}{\left(1-\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right)}\right)^k \tag{3.9}$$

Making use of the result in (3.9) above, equation (3.8) becomes

$$f(x) = \frac{\theta^2\left(\alpha e^{\beta x} e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)}{(1+\theta)\left(1-\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right)^3}\sum_{k=0}^{\infty}\frac{(-1)^k(\theta)^k}{k!}\left(\frac{\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)}{\left(1-\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right)}\right)^k$$

$$f(x) = \sum_{k=0}^{\infty}\frac{(-1)^k\theta^{k+2}\left(\alpha e^{\beta x} e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)}{k!(1+\theta)}\frac{\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^k}{\left(1-\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right)^{k+3}}$$

$$f(x) = \sum_{k=0}^{\infty}\frac{(-1)^k\theta^{k+2}\left(\alpha e^{\beta x} e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)}{k!(1+\theta)}\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^k\left(1-\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right)^{-(k+3)} \tag{3.10}$$

Also, using the generalized binomial theorem, we can write the last term from the above result as:

$$\left(1-\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\right)^{-(k+3)} = \sum_{i=0}^{\infty}\frac{\Gamma(i+k+3)}{i!\Gamma(k+3)}\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^i \tag{3.11}$$

Again making use of the expansion in (3.11) above, equation (3.10) can be written as:

$$f(x)=\sum_{i,k=0}^{\infty}\frac{(-1)^{k}\theta^{k+2}\Gamma(i+k+3)\left(\alpha e^{\beta x}e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)}{i!k!(1+\theta)\Gamma(k+3)}\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^{i+k} \tag{3.12}$$

Using the power series expansion in the above result, we have

$$\left(1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^{i+k}=\sum_{l=0}^{\infty}(-1)^{l}\binom{i+k}{l}e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)l} \tag{3.13}$$

$$f(x)=\sum_{i,k=0}^{\infty}\sum_{l=0}^{\infty}\binom{i+k}{l}\frac{(-1)^{k+l}\theta^{k+2}\Gamma(i+k+3)\alpha e^{\beta x}e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)l}}{i!k!(1+\theta)\Gamma(k+3)}$$

$$f(x)=\sum_{i,k=0}^{\infty}\sum_{l=0}^{\infty}\binom{i+k}{l}\frac{(-1)^{k+l}\theta^{k+2}\Gamma(i+k+3)\alpha e^{(1+l)\frac{\alpha}{\beta}}e^{\beta x}e^{-(1+l)\frac{\alpha}{\beta}e^{\beta x}}}{i!k!(1+\theta)\Gamma(k+3)} \tag{3.14}$$

Using power series expansion on the last term in the numerator part of equation (3.14), we have:

$$e^{-(1+l)\frac{\alpha}{\beta}e^{\beta x}}=\sum_{m=0}^{\infty}\frac{\left(-(1+l)\frac{\alpha}{\beta}\right)^{m}}{m!}e^{m\beta x} \tag{3.15}$$

Now, substituting equation (3.15), the power series expansion in equation (3.14) above, one gets:

$$f(x)=\sum_{i,k=0}^{\infty}\sum_{l,m=0}^{\infty}\binom{i+k}{l}\frac{(-1)^{k+l+m}\theta^{k+2}\Gamma(i+k+3)\left(\alpha(1+l)\right)^{m}\alpha e^{(1+l)\frac{\alpha}{\beta}}e^{\beta x}e^{m\beta x}}{i!k!m!\beta^{m}(1+\theta)\Gamma(k+3)}$$

$$f(x)=\sum_{i,k=0}^{\infty}\sum_{l,m=0}^{\infty}\binom{i+k}{l}\frac{(-1)^{k+l+m}\theta^{k+2}\Gamma(i+k+3)\left(\alpha(1+l)\right)^{m}\alpha e^{(1+l)\frac{\alpha}{\beta}}e^{\beta(1+m)x}}{i!k!m!\beta^{m}(1+\theta)\Gamma(k+3)} \tag{3.16}$$

Simplifying (3.16) above results in the following:

$$f(x)=\sum_{i,k=0}^{\infty}\sum_{l,m=0}^{\infty}\binom{i+k}{l}\frac{(-1)^{k+l+m}\theta^{k+2}\Gamma(i+k+3)\left(\alpha(1+l)\right)^{m}\alpha e^{(1+l)\frac{\alpha}{\beta}}e^{\beta(1+m)x}}{i!k!m!\beta^{m}(1+\theta)\Gamma(k+3)}$$

$$f(x)=\eta_{i,k,l,m}e^{\beta(1+m)x} \tag{3.17}$$

Where

$$\eta_{i,k,l,m}=\sum_{i,k=0}^{\infty}\sum_{l,m=0}^{\infty}\binom{i+k}{l}\frac{(-1)^{k+l+m}\theta^{k+2}\Gamma(i+k+3)\left(\alpha(1+l)\right)^{m}\alpha e^{(1+l)\frac{\alpha}{\beta}}}{i!k!m!\beta^{m}(1+\theta)\Gamma(k+3)}$$

Hence,

$$\mu_{n}^{'}=E\left[X^{n}\right]=\int_{0}^{\infty}x^{n}f(x)dx=\int_{0}^{\infty}\eta_{i,k,l,m}x^{n}e^{\beta(1+m)x}dx \tag{3.18}$$

Also, using integration by substitution method in equation (3.18); we obtain the following:

Let $-u = \beta(1+m)x \Rightarrow x = -\frac{u}{\beta(1+m)}$

$$-\frac{du}{dx} = \beta(1+m)$$

$$dx = \frac{-du}{\beta(1+m)x}$$

Substituting for u and dx in equation (3.18) and simplifying; we have:

$$\mu_n^{'} = E\left[X^n\right] = \int_0^\infty x^n f(x)dx = \eta_{i,k,l,m}\left[\frac{-1}{\beta(1+m)}\right]^{n+1}\int_0^\infty u^{n+1-1}e^{-u}du \tag{3.19}$$

Again recall that $\int_0^\infty t^{k-1}e^{-t}dt = \Gamma(k)$ and that $\int_0^\infty t^k e^{-t}dt = \int_0^\infty t^{k+1-1}e^{-t}dt = \Gamma(k+1)$

Thus we obtain the n^{th} ordinary moment of X for the Odd Lindley-Gompertz distribution as follows:

$$\mu_n^{'} = E\left[X^n\right] = \eta_{i,k,l,m}\left[\frac{-1}{\beta(1+m)}\right]^{n+1}\Gamma(n+1) \tag{3.20}$$

The Mean

The mean of the *OLnGD* can be obtained from the n^{th} moment of the distribution when *n=1* as follows:

$$\mu_n^{'} = E\left[X^n\right] = \eta_{i,k,l,m}\left[\frac{-1}{\beta(1+m)}\right]^{n+1}\Gamma(n+1)$$

$$\mu_1^{'} = E[X] = \frac{\eta_{i,k,l,m}}{\left[\beta(1+m)\right]^2} \tag{3.21}$$

Also the second moment of the *OLnGD* is obtained from the n^{th} moment of the distribution when *n=2* as

$$E\left[X^2\right] = \frac{-2\eta_{i,k,l,m}}{\left[\beta(1+m)\right]^3} \tag{3.22}$$

The Variance

The n^{th} central moment or moment about the mean of *X*, say μ_n, can be obtained as

$$\mu_n = E\left[X - \mu_1^{'}\right]^n = \sum_{i=0}^n (-1)^i \binom{n}{i} \mu_1^{'i}\mu_{n-i}^{'} \tag{3.23}$$

The variance of *X* for *OLnGD* is obtained from the central moment when *n=2,* that is,

$$Var(X) = E[X^2] - \{E[X]\}^2 \quad (3.24)$$

$$Var(X) = \frac{-2\eta_{i,k,l,m}}{\left[\beta(1+m)\right]^3} - \left\{\frac{\eta_{i,k,l,m}}{\left[\beta(1+m)\right]^2}\right\}^2 \tag{3.25}$$

The variation, skewness and kurtosis measures can also be calculated from the non-central moments using some well-known relationships.

3.2.2 Moment Generating Function

The moment generating function (*mgf*) is a simple way of arranging all the respective moments in a single function. It produces all the moments of the random variable by way of differentiation i.e., for any real number say *k*, the k^{th} derivative of □□(□) evaluated at $t = 0$ is the kth moment μ_k' of X.

The *mgf* of a random variable *X* can be obtained by

$$M_x(t) = E\left[e^{tx}\right] = \int_0^\infty e^{tx} f(x)dx \tag{3.26}$$

Recall that by power series expansion,

$$e^{tx} = \sum_{n=0}^{\infty}\frac{(tx)^n}{n!} = \sum_{n=0}^{\infty}\frac{t^n}{n!}x^n \tag{3.27}$$

Using the result in equation (3.27) and simplifying the integral in (3.26), therefore we have;

$$M_x(t) = E\left[e^{tx}\right] = \sum_{n=0}^{\infty}\frac{(tx)^n}{n!} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\mu_n'$$
$$M_x(t) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\left\{\eta_{i,k,l,m}\left[\frac{-1}{\beta(1+m)}\right]^{n+1}\Gamma(n+1)\right\} \tag{3.28}$$

Where *n* and *t* are constants, *t* is a real number and μ_n' denotes the n^{th} ordinary moment of *X* and equation (3.28) is the moment generating function of the Odd Lindley-Gompertz distribution.

3.2.3 Characteristics Function

The characteristics function has many useful and important properties which give it a central role in statistical theory. Its approach is particularly useful for generating moments, characterization of distributions and in analysis of linear combination of independent random variables.

The characteristics function of a random variable *X* is given by;

$$\varphi_x(t) = E\left[e^{itx}\right] = E\left[\cos(tx) + i\sin(tx)\right] = E\left[\cos(tx)\right] + E\left[i\sin(tx)\right] \tag{3.29}$$

Recall from power series expansion that

$$\cos(tx) = \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!}x^{2n}$$

$$E\left[\cos(tx)\right] = \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!}\mu_{2n}'$$

And also that

$$\sin(tx) = \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}x^{2n+1}$$

$$E\left[\sin(tx)\right] = \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}\mu_{2n+1}'$$

Simple algebra and power series expansion proves that

$$\phi_x(t)=\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!}\mu'_{2n}+i\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}\mu'_{2n+1} \quad (3.30)$$

Where μ'_{2n} and μ'_{2n+1} are the moments of X for n=2n and n=2n+1 respectively and can be obtained from μ'_n in equation (3.20)

3.3 Order Statistics

Sample values such as the smallest, largest, or middle observation from a random sample provide important information. For example, the highest rainfall, flood or minimum temperature recorded during past years might be useful when planning for future emergencies. Let $X_{(1)}$ denote the smallest of $X_1, X_2,...., X_n$, $X_{(2)}$ denote the second smallest of $X_1, X_2,...., X_n$, and similarly $X_{(i)}$ denote the i^{th} smallest of $X_1, X_2,...., X_n$.Then the random variables $X_{(1)}, X_{(2)},...., X_{(n)}$, called the order statistics of the sample $X_1, X_2,...., X_n$, has probability density function of the i^{th} order statistic, $X_{(i)}$, as:

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}f(x)F(x)^{i-1}\left[1-F(x)\right]^{n-i} \quad (3.31)$$

Where $f(x)$ and $F(x)$ are the *pdf* and *cdf* of the *OLnGD* respectively.

Using (3.5) and (3.6), the *pdf* of the i^{th} order statistics $X_{i:n}$, can be expressed from (3.27) as;

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}\left[\frac{\alpha\theta^2 e^{\beta x}e^{2\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}{(1+\theta)}\exp\left\{-\theta\left[e^{\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}-1\right]\right\}\right]$$
$$\left[1-\frac{\theta+e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}{(1+\theta)e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}\exp\left\{-\theta\left[e^{\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}-1\right]\right\}\right]^{i+k-1} \quad (3.32)$$

Hence, the *pdf* of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the *OLnGD* are given by;

$$f_{1:n}(x)=n\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\left[\frac{\alpha\theta^2 e^{\beta x}e^{2\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}{(1+\theta)}\exp\left\{-\theta\left[e^{\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}-1\right]\right\}\right]$$
$$\left[1-\frac{\theta+e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}{(1+\theta)e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}\exp\left\{-\theta\left[e^{\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}-1\right]\right\}\right]^{k} \quad (3.33)$$

and

$$f_{n:n}(x)=n\left[\frac{\alpha\theta^2 e^{\beta x}e^{2\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}{(1+\theta)}\exp\left\{-\theta\left[e^{\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}-1\right]\right\}\right]$$
$$\left[1-\frac{\theta+e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}{(1+\theta)e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}\exp\left\{-\theta\left[e^{\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}-1\right]\right\}\right]^{n-1} \quad (3.34)$$

respectively.

3.5 Estimation of Parameters

Let $X_{1, - - -,} X_n$ be a sample of size 'n' independently and identically distributed random variables from the *OLnGD* with unknown parameters *α, β* and *Θ* defined previously. The *pdf* of the *OLnGD* is given as:

$$f(x)=\frac{\alpha\theta^2 e^{\beta x} e^{2\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}{(1+\theta)}\exp\left\{-\theta\left[e^{\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}-1\right]\right\}$$

The likelihood function is given by;

$$L\left(X_1,X_2,.....,X_n/\theta,\alpha,\beta\right)=\left(\frac{\alpha\theta^2}{1+\theta}\right)^n e^{\beta\sum_{i=1}^{n}x_i} e^{2\frac{\alpha}{\beta}\sum_{i=1}^{n}\left(e^{\beta x_i}-1\right)} e^{-\theta\sum_{i=1}^{n}\left(e^{\frac{\alpha}{\beta}\left(e^{\beta x_i}-1\right)}-1\right)} \tag{3.35}$$

Taking the natural logarithm of the likelihood function, i.e., Let, $l(n)=\log L\left(X_1,X_2,.....,X_n/\theta,\alpha,\beta\right)$, such that

$$l(n)=n\log\alpha+2n\log\theta+\beta\sum_{i=1}^{n}x_i-n\log(1+\theta)+\tfrac{2\alpha}{\beta}\sum_{i=1}^{n}\left(e^{\beta x_i}-1\right)-\theta\sum_{i=1}^{n}\left(e^{\frac{\alpha}{\beta}\left(e^{\beta x_i}-1\right)}-1\right) \tag{3.36}$$

Differentiating $l(n)$ partially with respect to Θ, α and β respectively gives;

$$\frac{\partial l(n)}{\partial\theta}=\frac{2n}{\theta}-\frac{n}{\theta+1}-\sum_{i=1}^{n}\left(e^{\frac{\alpha}{\beta}\left(e^{\beta x_i}-1\right)}-1\right) \tag{3.37}$$

$$\frac{\partial l(n)}{\partial\alpha}=\frac{n}{\alpha}+\tfrac{2}{\beta}\sum_{i=1}^{n}\left(e^{\beta x_i}-1\right)-\tfrac{\theta}{\beta}\sum_{i=1}^{n}\left(e^{\beta x_i}-1\right)e^{\frac{\alpha}{\beta}\left(e^{\beta x_i}-1\right)} \tag{3.38}$$

$$\frac{\partial l(n)}{\partial\beta}=\sum_{i=1}^{n}x_i+\tfrac{2}{\beta}\sum_{i=1}^{n}x_i e^{\beta x_i}-\tfrac{2}{\beta^2}\sum_{i=1}^{n}\left(e^{\beta x_i}-1\right)+\tfrac{\alpha\theta}{\beta}\sum_{i=1}^{n}x_i e^{\beta x_i}e^{\frac{\alpha}{\beta}\left(e^{\beta x_i}-1\right)}-\tfrac{\alpha\theta}{\beta^2}\sum_{i=1}^{n}\left(e^{\beta x_i}-1\right)e^{\frac{\alpha}{\beta}\left(e^{\beta x_i}-1\right)} \tag{3.39}$$

Equating equations (3.37), (3.38) and (3.39) to zero and solving for the solution of the non-linear system of equations will give us the maximum likelihood estimates of parameters θ,α,and,β respectively. However, the solution cannot be obtained analytically except numerically with the aid of suitable statistical software like Python, R, SAS, e.t.c when data sets are given.

3.6 Conclusion

In this paper, we introduced and studied some mathematical and statistical properties of a new distribution, the Generalized Odd lindley Gompertz Distribution (GOGD). We have derived explicit expression for its survival function, order statistics and ordinary moment. It was found that the GOGD has various shape patterns depending on the parameter values. For example, negatively skewed with a higher degree of kurtosis. Some plots for the cdf and pdf of the OLGD show that the GOGD can be used to model variables whose chances of success in a given interval decreases with increase in time whereas that of failure increases as time increases i.e. it has an increasing failure rate useful for modeling lifetime data. We also obtained the pdf of its minimum and maximum order statistics. The estimation of the model parameters is being done using the method of maximum likelihood estimation.

Competing Interests

The authors have declared that they have no competing interest exist.

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