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RESEARCH ARTICLE

ON THE HYPERSURFACE OF RANDERS CHANGE OF FINSLER SQUARE METRIC

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ABSTRACT

In this paper, we introduce a special Finsler metric that is Randers change Square metric. Then we studied the criterion for a hypersurface of a special Finsler metric to be a hyper plane of the first kind, the second kind but not of third kind.

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INTRODUCTION

The theory ofhyper surface of Finsler spaces may be developed after the model of Riemanniangeometry. In the early years of Finsler geometry, E.Cartan, made M. Haimovici fundamental and essential contributions to the theory. Then, various interesting results on hyper surface of Finsler spaces have been found by O. Varga, H.Rund (see.(1),(3),(4),(7),(8)).It seems, however, to be the un evitable obstructions to develop the theory of hyper surface of Finsler spaces analogously to the Riemannian theory, and, as a consequence, almost all the existing literatures are not easy to understand and confused notations sometimes bewilder the readers. The first among those obstructions is perhaps surviving of quantities which are derived from Cartan's C-tensor, for instance, the non-symmetry property of the second fundamental tensor. The second consequence ofthe first, is that the induced connection, defined by the projection, does not generally coincide with the intrinsic connection, determined from the induced Finsler metric, and that the former is beyond the usual concept of connection appearing in Finsler geometry. In 1985, Matsumoto (9) studied the theory of Finslerian hyper surfaces and various types of Finslerian hyper surfaces called hyper planes of the first kind, second kind and third kind. In 1972, Makoto Matsumoto proposed the concept of (α, β)- metric, where $\alpha = a_{ij}(x^i y^j)^{\frac{1}{2}}$ is a Riemannian metric and $\beta = bi(x)y^i$ is a 1-form on a smooth manifold. He induced the intrinsic Finsler connections of a hypersurface of the Finsler space. M. Hashiguchi and Y. Ichi-jyo and C. Shibata were studied Finsler space with differerent (α, β) - metrics. In 1992, Makoto Matsumoto also worked on the theory of Finsler spaces with (α, β)- metric and obtained the interesting results. In 1980 H.Wosoughi has studied the theory of hypersurface of special Finsler space with an Exponential (α, β) -metric . L.Y. Lee. H.Y. Park and Y.D. Lee were got the some results on hypersurface of a special Finsler space with a metric $+\frac{\beta^2}{\alpha}$. In 2008 M.K. Gupta and P.N. Pandey studied the hypersurface of the Finsler space equipped with a Randers conformal metric and obtained certain geometrical properties of the hypersurface of the Finsler space. In 2009 S.K.Narasimhamurthy and H.G.Nagaraj were studies the Special hypersurface of a Finsler space with (α, β)- metrics. A change of Finsler metric $L(x;\, y)\bar{L} = L(x;\, y) + bi(x)y^i$ is called Randers change of metric.

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The notion of a Hashiguchi and Ichijyo and studied in detail by Shibata. Recently Nagaraja and Kumar studied the properties of a Finsler space with the Randers chage of Matsumoto metric. In this paper , we syudy the hypersurface of a special Finsler space Fn equipped with the metric function $F = \frac{(\alpha+\beta)^2}{\alpha} + \beta$ and here by inspiring the work of On the Hyper surface of a Finsler space with the Square metric and to obtain necessary and sufficient conditions for the hyper surface to be a hyper plane of the first kind ,the second kind but not the third kind .

Preliminaries: A Finsler manifold is a manifold M and a function $F : TM \rightarrow [0,\infty)$ called a Finsler norm such that

- F is smooth on $TM\backslash\{0\}$.
- $F|TxM : TxM \rightarrow [0,\infty)$ is a Minkowski norm for all $x \in M$.

Here, $TM\backslash\{0\}$. is the slashed tangent bundle. i.e., $TM\backslash\{0\}$. $= \cup \{TxM\backslash 0\} : x \in M$.
Example: Let (M, g) be a Riemannian manifold. Then $F(x,y) = g_x(y.y)$ is a Finsler norm on M.

Let $Fn = (Mn, L)$ be an n-dimensional Finsler space equipped with the fundamental func-tion $L(x,y)$ satisfying the requisite conditions. The normalized supporting element, the metric tensor, the angular metric tensor and Cartan tensor defined by

$$li = \dot{\partial}_i L \quad g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2 \quad hij = L\dot{\partial}_i \dot{\partial}_j \quad C_{ijk} = \frac{1}{2} \dot{\partial}_k g_{ij}$$

respectively. The inverse of the metric tensor and the generalized christoffe l symbols are given by,

$$g_{ij} g^{jk} = \delta k_i,$$
$$\gamma_{ijk} = \frac{1}{2}\{\partial_i g_{jk} + \partial_k g_{ij} - \partial_j g_{ik}\},$$
$$\gamma^i_{jk} = g^{ir} \gamma_{jrk}.$$

The Cartan connection in Fn is given as

$$C\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$$

Where

$$F^i_{jk} = \frac{1}{2}\{\delta_i g_{jk} + \delta_k g_{ij} - \delta_j g_{ik}\}, \tag{1}$$

$$G^i_j = \dot{\partial}_j G^i \,, G^i = \frac{1}{2} \gamma^i_{jk} y^j y^k \,, \quad C^i_{jk} = g^{ir} c_{jrk}. \tag{2}$$

Here, we shall use the following definitions, which are defined by M.Matstumoto in (9);

1. Definition: If each path of a hyper surface F^{n-1} with respect to the induced connection is also a path of the enveloping space F^n then F^{n-1} is called a hyper plane of the first kind.
2. Definition: If each path of a hyper surface F^{n-1} with respect to the induced connection is also a h- path of the enveloping space F^n , then F^{n-1} is called a hyper plane of the second kind.
3. Defnition: If the unit normal vector of F^{n-1} is parallel along each curve of F^{n-1} ; then F^{n-1} is called a hyper plane of the third kind.

Notion of Randers change of Square Finslermetric: Consider the Finsler metric (M^n, F), where F is the Randers change Finsler Square metric that is given by

$$F = \frac{(\alpha+\beta)^2}{\alpha} + \beta \tag{3}$$

Differentiating equation (3.1) partially with respect to α and β twisely, we get

$$F_\alpha = \frac{\alpha^2-\beta^2}{\alpha^2},\ F_{\alpha\alpha} = \frac{2\beta^2}{\alpha^3},\ F_\beta = \frac{3\alpha+2\beta+\alpha}{\alpha},\ F_{\beta\beta} = \frac{2}{\alpha},\ F_{\alpha\beta} = \frac{-2\beta^2}{\alpha}. \tag{4}$$

The angular metric tensor hij is given by

$$h_{\alpha\beta} = pa_{ij} + q_0\, b_i b_j + q_1(b_i y_j + b_j y_i) + q_2 y_i y_j\,, \tag{5}$$

where the coefficients are defined and calculated as follows

$$y_i = a_{ij} y^j,$$
$$l_i = \frac{F_\alpha y_i}{\alpha} + F_\beta b_i\,, \tag{6}$$

$$p = \frac{FF_\alpha}{\alpha} = \frac{((\alpha+\beta)^2+\alpha\beta)(\alpha^2-\beta^2)}{\alpha^4}\,, \tag{7}$$

$$q_1 = \frac{FF_{\alpha\beta}}{\alpha} = \frac{-2\beta(\alpha+\beta)^2+\alpha\beta}{\alpha^4}\,, \tag{8}$$

$$q_2 = \frac{F(F_{\alpha\alpha}-\frac{F_\alpha}{\alpha})}{\alpha^2} = \frac{2\beta^2-(\alpha^2-\beta^2)}{\alpha^4}. \tag{9}$$

The fundamental metric tensor $g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$ of Finsler space (M^n, F) is defined by

$$g_{ij} = pa_{ij} + p_0\, b_i b_j + p_1(b_i y_j + b_j y_i) + p_2 y_i y_j\,, \tag{10}$$

where the coefficients p, p_0, p_1 and p_2 are defined and calculated as follows

$$p = \frac{FF_\alpha}{\alpha} = \frac{((\alpha+\beta)^2+\alpha\beta)(\alpha^2-\beta^2)}{\alpha^4}\,, \tag{11}$$

$$p_0 = q_0 + F_\beta^2 = 2 + \frac{(3\alpha+2\beta+\alpha)^2}{\alpha^2}\,, \tag{12}$$

$$q_0 = F_{\beta\beta}\left\{\frac{(\alpha+\beta)^2}{\alpha} + \beta\right\}\left\{\frac{2}{\alpha}\right\}, \tag{13}$$

$$p_1 = q_1 + \frac{pF_\beta}{F} = \left\{\frac{-2\beta(\alpha+\beta)^2+\alpha\beta}{\alpha^4}\right\}\left\{\frac{(\alpha+\beta)^2}{\alpha} + \beta\right\}\left\{\frac{\alpha^2-\beta^2}{\alpha^2}\right\}\frac{(3\alpha+2\beta+\alpha)}{\{(\alpha+\beta)^2+\alpha\beta\}}, \tag{14}$$

$$p_2 = q_2 + \frac{p^2}{F^2} = \left\{\frac{2\beta^2-(\alpha^2-\beta^2)}{\alpha^4}\right\} + \left\{\frac{[((\alpha+\beta)^2+\alpha\beta\ (\alpha^2-\beta^2)]^2(\alpha+\beta)^2+\alpha\beta}{\alpha^6}\right\}. \tag{15}$$

In addition, reciprocal tensor g^{ij} of fundamental metric tensor of Finsler space F^n is

$$g^{ij} = \frac{a_{ij}}{p} - S_0 b^i b^j - S_1(b^i y^j + b^j y^i) - S_2 y^i y^j\,, \tag{16}$$

where the coefficients $b_i, S_0\ S_1$ and S_2 are defined as follows:

$$b^i = a^{ij} b_j\,,$$

$$S_0 = \frac{pp0 + (p0p2 - p_1^2)\alpha^2}{p\zeta}, \quad (17)$$

$$S_1 = \frac{pp1 + (p0p2 - p_1^2)\beta}{p\zeta}, \quad (18)$$

$$S_2 = \frac{pp2 + (p0p2 - p_1^2)b^2}{p\zeta}, \quad (19)$$

$$\zeta = p(p + p_0 b^2 + p_1\beta) + (p_0 p_2 - p_1^2)(\alpha^2 b^2 - \beta^2), \quad (20)$$

where $b^2 = a_{ij} b^i b^j$.

Now, the hv-torison tensor $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ is defined by

$$C_{ijk} = \frac{p(h_{ij}m_k = h_{jk}m_i + h_{ki}m_j) + \gamma_1 m_i m_j m_k}{2p} \quad (21)$$

where the coefficients γ_1 and m_i are defined as

$$\gamma_1 = p\frac{\partial p_0}{\partial \beta} - 3p_1 q_0, \quad m_i = b_i - \frac{y_i \beta}{\alpha^2}, \quad (22)$$

where $p0, p, p1$ are given in $(11), (12), (14)$ and m_i is known as non-zero covariant vector orthogonal to supporting element y^i

Let Γ_{jk}^i be the components of Christoffel symbol of the associated Riemannian space R^n and ∇_k be the covariant differentiation with respect to x^k relative to Christoffel symbol.

Now we put

$$r_{ij} = \frac{1}{2}(b_{ij} + b_{ji}), \quad (23)$$

$$S_{ij} = \frac{1}{2}(b_{ij} + b_{ji}) \quad (24)$$

where $b_{ij} = \nabla_j b_i$.

Let $C\Gamma = (\Gamma_{jk}^{*i}, \Gamma_{0k}^{*i}, G_{jk}^i)$ be Cartan connection of F^n. The difference tensor $D_{jk}^i - \Gamma_{jk}^i$ of the special Finsler space F^n is given by

$$D_{jk}^i = B^i r_{jk} + s_j^i B_k + B_j^i b_{0j} - b_{0m} g^{im} B_{jk} - C_{jm}^i A_k^m$$

$$-C_{km}^i A_j^m + C_{jkm} A_s^m g^{is} + \gamma^s (C_{jm}^i Csk^m + C_{km}^i C_{sj}^m - C_{jk}^m C_{ms}^i), \quad (25)$$

where $B_k = p_0 b_k + p_1 y_k$. (26)

Using (12) and (14) in (26) we get

$$B_k = 2 + \left\{\frac{(3\alpha + 2\beta + \alpha)^2}{\alpha^2}\right\} b_k - \left\{\frac{(2\beta(\alpha+\beta)^2 + \alpha\beta)((\alpha+\beta)^2 + \alpha\beta)(\alpha^2 - \beta^2)(3\alpha + 2\beta + \alpha)}{\alpha^3(\alpha+\beta)^2 + \alpha\beta}\right\} y_k \quad (27)$$

$$B^i = g^{ij} B_j, \quad (28)$$

$$B_{ij} = \frac{p_1\left(a_{ij} - \frac{y_i y_j}{\alpha^2}\right) + \frac{\partial p_0}{\partial \beta} m_i m_j}{2}, \quad (29)$$

$$2B_{ij} = -\left\{\frac{(2\beta(\alpha+\beta)^2+\alpha\beta)}{\alpha^4}\right\} + \left\{\frac{((\alpha+\beta)^2+\alpha\beta)(\alpha^2-\beta^2)(3\alpha+2\beta+\alpha)(a_{ij}\alpha^2-y_i y_j)}{\alpha^6}\right\}$$
$$+\left\{\frac{\alpha^2+4\alpha+2\beta}{\alpha^2}\right\} m_i m_j \quad (30)$$

$$B_i^k = g^{kj} B_{ji}. \quad (31)$$

$$A_k^m = B_k^m r_{00} + B^m r_{k0} + B_k s_0^m + B_0 s_k^m. \quad (32)$$

Using (26) in (32) we have

$$A_k^m = B_k^m r_{00} + B^m r_{k0} + 2 + \left\{\frac{(3\alpha+2\beta+\alpha)^2}{\alpha^2}\right\} b_k -$$

$$\left\{\frac{((\alpha+\beta)^2+\alpha\beta)(\alpha^2-\beta^2)(3\alpha+2\beta+\alpha)}{\alpha^3(\alpha+\beta)^2+\alpha\beta}\right\} y_k s_0^m + B_0 s_k^m. \quad (33)$$

$$\lambda^m = B^m r_{00} + 2B_0 s_0^m \quad (34)$$

$$s_i^k = g^{kj} s_{ji} \quad (35)$$

$$B_0 = B_i Y^i \quad (36)$$

Using (27) in above we get,

$$B_k = \left\{2 + \left\{\frac{(3\alpha+2\beta+\alpha)^2}{\alpha^2}\right\}\right\} b_i$$

$$-\left\{\left\{\frac{(2\beta(\alpha+\beta)^2+\alpha\beta)((\alpha+\beta)^2+\alpha\beta)(\alpha^2-\beta^2)(3\alpha+2\beta+\alpha)}{\alpha^3((\alpha+\beta)^2+\alpha\beta)}\right\} y_i\right\} Y^i \quad (37)$$

Induced Car tan Connection:

$Let\ F^n = (M^n, F)$, where $F = \frac{(\alpha+\beta)^2}{\alpha} + \beta$, be a Finsler space and let F^{n-1} be its. Hyper surface having equation $x^i = x^i(u^\alpha), i = 1,2, \ldots (n-1)$. Let the matrix of projection factor be $B_\alpha^i = \frac{\partial x^j}{\partial u^\alpha}$ and its rank is $(n-1)$. The tangential component of element of support y^i of Finsler space F^n along its hypersurface F^{n-1} is given by

$$y^i = B_\alpha^i(u) v^\alpha. \quad (38)$$

Therefore, v^α is the element of support of hypersurface F^{n-1} at the point u^α. The metric tensor $g_{\alpha\beta}$ and hv-torison tensor $C_{\alpha\beta\gamma}$ of hypersurface F^{n-1} are defined by

$$g_{\alpha\beta} = g_{ij} B_\alpha^i B\beta^j, \ C_{\alpha\beta\gamma} = C_{ijk} B_\alpha^i B_\beta^j B_\gamma^k \quad (39)$$

Now the unit normal vector N^i(u,v) at an arbitrary point u^α of the hypersurface F^{n-1} is definedby the following property;

$$g_{ij}\big(x(u,v),y(u,v)\big)B^i_\alpha N^j = 0\,,\; g_{ij}\big(x(u,v),y(u,v)\big)N^iN^j = 1 \tag{40}$$

The angular metric tensor h_{ij} , we have

$$h_{\alpha\beta} = h_{ij}B^i_\alpha B^j_\beta\,,\quad h_{ij}B^i_\alpha N^j = 0\,,\quad h_{ij}N^iN^j = 1. \tag{41}$$

Let $(B^i_\alpha N_i)$ be the inverse of (B^i_α, N^i) , then we have

$$B^i_\alpha = g^{\alpha\beta}g_{ij}\,B^j_\beta\,, B^i_\alpha\,B^\beta_i = \delta^\beta_\alpha\,,\quad B^i_\alpha\,N^i = 0\,, \tag{42}$$

$$B^i_\alpha N_i = 0\,, N_i = g_{ij}\,N^i\,, B^k_i g^{kj} B_{ji}\,, \tag{43}$$

$$B^i_\alpha\,B^\alpha_j + N^iN_j = \delta^i_j\,, \tag{44}$$

The induced Cartan connection $\pounds C\Gamma = (\Gamma^{*\alpha}_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})$ on hypersurface F^{n-1} induced from the Cartan's connection $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^i_{jk})$ is given by (3);

$$\Gamma^{*\alpha}_{\beta\gamma} = B^\alpha_i\left(B^i_{\beta\gamma} + \Gamma^{*i}_{jk}\,B^j_\beta B^k_\gamma\right) + M^\alpha_\beta H_\gamma, \tag{45}$$

$$G^\alpha = B^\alpha_i\left(B^i_{0\beta} + \Gamma^{*i}_{0j}\;B^j_\beta\right), \tag{46}$$

$$C^\alpha_{\beta\gamma} = B^\alpha_i\;C^i_{jk}\;B^j_\beta B^k_\gamma\,, \tag{47}$$

where $M_{\beta\gamma} = N_iC^i_{jk}\;B^j_\beta\,\gamma^k$ (48)

$$M^\alpha_\beta = g^{\alpha\gamma}M_{\beta\gamma}\,, \tag{49}$$

$H_\beta = N_i(B^i_{0\beta} + \Gamma^{*i}_{0j}\;B^j_\beta)$, (50)

$B^i_{\beta\gamma} = \frac{\partial B^i_\beta}{\partial u^\gamma}$,(51)

$B^i_{0\beta} = b^i_{\alpha\beta}v^\alpha$. (52)

The quantities $M_{\beta\gamma}$ and H_β appeared in above equation are called the second fundamental $v-$tensor $H_{\beta\alpha}$ is defined as (8);

$$H_{\beta\gamma} = N_i\left(B^i_{\beta\gamma} + \Gamma^{*i}_{jk}\,B^j_\beta B^k_\gamma\right) + M_\beta H_\gamma\,, \tag{53}$$

$$M_\beta = N_iC^i_{jk}\;B^j_\beta N^k\,. \tag{54}$$

The relative $h-$ covariant derivative and $v-$covariant derivative of projection factor B^α_i with respect to induced Cartan connection $\pounds C\Gamma$ are respectively given by;

$$B^i_{\alpha|\beta} = H_{\alpha\beta}N^i. \tag{55}$$

$$B^i_{\alpha|\beta} = M_{\alpha\beta}N^i . \tag{56}$$

The equation (3.10) shows that $H_{\beta\gamma}$ is not always symmetric and

$$H_{\beta\gamma} - H_{\gamma\beta} = M_\beta H_\gamma - M_\gamma H_\beta . \tag{57}$$

Thus the above equation simplifies to

$$H_{0\gamma} = H_\gamma , \; H_{\gamma 0} = H_\gamma + M_\gamma H_0 . \tag{58}$$

We utilize the following lemma derived by M.Matsumoto (9) in order to prove our hypothesis. Lemma 1. The normal curvature $H_0 = H_\beta v^\beta$ vanishes if and only if normal curvature vector H_β vanishes. Lemma 2.A hyper surface F^{n-1} is a hyperplane of first kind if and only if $H_\alpha = 0$. Lemma 3.A hyper surface F^{n-1} is a hyperplane of second kind with respect to Cartan connection $C\Gamma$ if and only if $H_\alpha = 0 \; and \; H_{\alpha\beta} = 0$.

Lemma 4.A hyper surface F^{n-1} is a hyperplane of third kind with respect to Cartan connection $C\Gamma$ if and only if H_α , $H_{\alpha\beta} = 0$ and $M_{\alpha\beta} = 0$.

Hyper surface of Randers change of Square metric: This section determines the necessary and sufficient condition of F^n for hyper surface to be hyper plane is satisfied the first kind, second kind but not third kind.

Consider the hyper surface F^{n-1} whose equation is given by

$$b_i(x) = c , \tag{59}$$

where c is a fixed constant.

Thus, the gradient of the function representing hyper surface F^{n-1} is given by in tensor notation

$$b_i(x) = \partial_i b .$$

Again, consider the parametric equation $x_i(u^\alpha)$ of the hypersurface F^{n-1} .

Differentiating the equation (59) with respect to parameter u^α we get $\partial_\alpha b\big(x(u)\big) = 0 = b_i B^i_\alpha$. It is obvious that $bi(x)$ are the covariant component of normal vector field of hyper surface F^{n-1}.

Therefore along the hyper surface F^{n-1} we have

$$b_i B^i_\alpha = 0. \tag{60}$$

$$b_i y^i = 0. \tag{61}$$

The metric induced $L(u, v)$ from the Randers change of Finsler Square space (M^n, F).

Where $F = \frac{(\alpha+\beta)^2}{\alpha} + \beta$ on the hyper surface F^{n-1} is given by

$$L(u, v) = a_{\alpha\beta} u^\alpha v^\beta , \tag{62}$$

Where $a_{\alpha\beta} = a_{ij} B^i_\alpha B^j_\beta$.

The induced metric in (62) do not have β component $(i.e., \beta = b_i y^i = 0)$ of the Finsler metric of the original space (M^n, F), therefore induced metric in equation (62) is a Riemannian metric. Therefore at any point on the hyper surface F^{n-1} .

Equations (8),(9),(11), (12),(14) and (15) reduced to

$$p = 1, q_0 = 2, q_2 = -1, p_0 = 12, p_1 = 4, p_2 = 0. \quad (63)$$

$$s_0 = \frac{-4}{1-4b^2}, s_1 = \frac{4}{1-4b^2}, s_2 = \frac{-16b^2}{1-4b^2}, \zeta = 1 - 4b^2. \quad (64)$$

Theorem 1. The induced metric from the Randers Change Square metric on the hyper surface F^{n-1} is a scalar function $b(x)$ is given by

$b_i(x(u)) = \frac{4b}{\sqrt{1-4b^2}} N_i$ and $b_i = a^{ij} b_j = \sqrt{1-4b^2} N_i + b^2 y^i$

where N_i is unit normal vector.

Proof: Using equation (63) and (64) in (16) we have

$$g^{ij} = a^{ij} + \frac{4}{1-4b^2} b^i b^j - \frac{4}{1-4b^2}(b^i y^j + b^j y^i) + \frac{16 b^2}{1-4b^2} y^i y^j. \quad (65)$$

Multiplying equation (4.7) by $bibj$ and we know that the fact that $\beta = b_i y^i = 0$, it becomes

$$g^{ij} b_i b_j = \frac{16 b^2}{1-4b^2}. \quad (66)$$

Thus we get

$$b_i(x(u)) = \frac{4b}{\sqrt{1-4b^2}} N_i, \quad b^2 = a^{ij} b_i b_j, \quad (67)$$

where b is the length of the vector bi.
Now, from (65) and (67) we get

$$b^i = a^{ij} = \sqrt{1-4b^2} N^i + b^2 y^i. \quad (68)$$

Theorem 2. If F be Randers change of Square metric on the Finsler hyper surface F^{n-1} then the second fundamental tensor is given by

$$M_{\alpha\beta} = \frac{1}{\alpha} \frac{4b}{\sqrt{1-4b^2}} h_{\alpha\beta} \text{ and } M_\alpha = 0.$$

And the second fundamental h- tensor $H_{\alpha\beta}$ is symmetric. Proof: Using (63) in (10) and (5), then we get fundamental metric tensor gij and angular metric tensor h_{ij}

$$g_{ij} = a_{ij} + 12 b_i b_j + 4(b_i y_j + b_j y_i) \quad (69)$$

$$h_{ij} = a_{ij} + 2 b_i b_j - y_i y_j. \quad (70)$$

From (15), (61) and (41) it follows that, if $h^{(a)}_{\alpha\beta}$ denotes the angular metric tensor of Riemannian $a_{ij}(x)$, then along hypersurface $F^{n-1}(c)$, $h_{\alpha\beta} = h^{(a)}_{\alpha\beta}$. Differentiate equation $p0$ with respect to β we have

$$\frac{\partial p_0}{\partial \beta} = \frac{\alpha^2 + 4\alpha + 2\beta}{\alpha^2}. \quad (71)$$

Thus, along the hyper surface $F^{n-1}(c)$ the equation (71) reduces to $\frac{\partial p_0}{\partial \beta} = (1 + \frac{4}{\alpha})$ and the equation γ_i, mi reduces to $\gamma_1 = -3$, $mi = bi$.

Now, using the values $of\ p, p1$, γ_1 and mi in (21), hv - torsion tensor in hyper surface F^{n-1} is

$$C_{ijk} = h_{ij} b_k + h_{jk} b_i + h_{ki} b_j , \quad (72)$$

using (60) and (72) in (45) we get

$$M_{\alpha\beta} = \frac{1}{\alpha} \frac{4b}{\sqrt{1-4b^2}} \quad (73)$$

From (60) and (72) in (51) we get

$$M_\alpha = 0. \quad (74)$$

Observing the above equation in (53), which shows that $H_{\alpha\beta} = H_{\beta\alpha}$ i.e., $H_{\alpha\beta}$ is symmetric.

Theorem3. The necessary and sufficient condition of Randers change Square metric for hypersurface $F^{n-1}(c)$ to be hyperplane of first kind equation is

$$b_{ij} = \frac{1}{2}(b_i c_j + b_j c_i)$$

and then the second fundamental tensor of $F^{n-1}(c)$ is proportional to its $h_{\alpha\beta}$. Proof: Differentiate (60) with respect to β we get

$b_{i|j} B^i_\alpha + b_i B^i_{\alpha|\beta} = 0$, (75)

using (3.50) and $b_{i|j} = B^j_\beta + b_{i|j} N^j H_\beta = 0$ equation (75) reduces to

$$b_{i|j}\, B^j_\beta B^i_\alpha + b_{i|j} N^j H_\beta B^i_\alpha + b_i H_{\alpha\beta} N^i = 0 , \quad (76)$$

Since $b_{i|j} = -b_h C^h_{ij}$, from (51), (67) and (74) we get

$$b_{i|j}\ B^i_\alpha N^j = \frac{4b}{\sqrt{1-4b^2}} M_\alpha = 0 , \quad (77)$$

using (76) in (77) we get

$$b_{i|j}\, B^j_\beta B^i_\alpha + \frac{4b}{\sqrt{1-4b^2}} H_{\alpha\beta} = 0. \quad (78)$$

Clearly, $b|j$ is symmetric. Now, contracting (78) with v^β first and again contracting with v^α respectively and then

using (38), (55) and (74) we get

$$b_{i|j}\ B^i_\alpha\, y^j + \frac{4b}{\sqrt{1-4b^2}} H_\alpha = 0 , \quad (79)$$

$$b_{i|j}\ y^i\, y^j + \frac{4b}{\sqrt{1-4b^2}} H_0 = 0 . \quad (80)$$

From the Lemma (4) and Lemma (5) a hyper surface $F^{n-1}(c)$ is a hypersurface of first kind iff normal curvature vanishes i.e., H0 = 0.

Since H0 = 0 in (80), we find that hyper surface $F^{n-1}(c)$ is again a hyperplane of first kindiff $b_{i|j}\, y^i y^j = 0$.

Note that: $b_{i|j}$ is the covariant derivative of with respect to Cartan connection $C\Gamma$ Finsler. Space F^n it may be depends on y^i. Moreover $\nabla_j b_i = b_{ij}$.Which is the covariant derivative with respect to Riemannian connection Γ^i_{jk} constructed from $a_{ij}(x)$ therefore b_{ij} does not depend on y^i.

On the other hand, consider the difference $b_{i|j} - b_{j|i}$ and the difference tensor

$$D^i_{jk} = \Gamma + \Gamma^{*i}_{jk} - \Gamma^i_{jk}$$

is given by (25). Since bi is a gradient vector from (23) and (24) we have

$$r_{ij} = b_{ij}\,, s_{ij} = 0\,, s^i_j = 0. \tag{81}$$

From (81) in to (25) we get

$$D^i_{jk} = B^i\, b_{jk} + s^i_k B_j + B^i_j b_{0j} - b_{0m} g^{im}\, B_{jk} - C^i_{jm} A^m_k - C^i_{km}\, A^m_j + C_{jkm} A^m_s g^{is} + \gamma^s\,(C^i_{jm}\, Csk^m + C^i_{km}\, C^m_{sj} - C^m_{jk} C^i_{ms}), \tag{82}$$

Now, using (8), (9) , (10) and (65) in to (26) to (60), we get

$$B_k = 6b_k + \frac{2}{\alpha} y_k\,, B^i = \frac{4b}{1-4b^2} b^i + \frac{2}{\alpha(1-4b^2)} y_i\,. \tag{83}$$

$$B_{ij} = \frac{4\alpha(\alpha^2 a_{ij} - y_i y_j) + (\alpha^2 + 4\alpha + 2\beta) 6 b_i b_j}{2\alpha^3} \tag{84}$$

$$B^i_j = \{a_{ij} + \frac{4}{1-4b^2}\, b^i b^j - \frac{4}{1-4b^2}\,(b^i y^j + b^j y^i) + \frac{16 b^2}{1-4b^2}\} B_{ji} \tag{85}$$

$$A^m_k = B^m_k b_{00} + B^m b_{k0}\,, \lambda^m = B^m b_{00}. \tag{86}$$

On contracting (86) by y^j we get

$$B_{i0} = 0\,, B^i_0 = 0\,.$$

Again contracting (86) by y^k and using $B^i_0 = 0$ we get

$$A^m_0 = B^m b_{00}.$$

Also, contracting (82) by y^k and using the facts

$$B_{i0} = 0\,, B^i_0 = 0\,. A^m_0 = B^m b_{00} \text{ and } C^m_{s0} = 0\,, C^i_{0m} = 0\,, C^m_{j0} = 0\,.$$

Then by cotracting (39) and (47) we get

$$D^i_{j0} = B^i b_{j0} + B^i_j b_{00} - b_{00} B^m C^i_{jmi}, \tag{87}$$

$$D^i_{00} = \frac{4b}{1-4b^2} b^i b_{00} + \frac{4b}{\alpha(1-4b^2)} y^i b_{00}. \tag{88}$$

On multiplying (87) by b^i and then using (61), (83), (86) and (87) we get

$$b_i D^i_{j0} = \frac{4b^2}{1-4b^2} b_{j0} + \frac{1-12b^2}{\alpha(1-4b^2)} b_j b_{00} - \frac{4b}{1-4b^2} b^i\, b^m\, C^i_{jm} b_{00}\,. \tag{89}$$

Now, multiplying (88) by b^i and then using (61) we get

$$b_i D^i_{00} = \frac{4b^2}{1-4b^2}\, b_{00}\,. \tag{90}$$

From (72) it is clear that

$$b^m b_i C^i_{jm} B^j_\alpha = b^2 M_\alpha = 0\,.$$

On contracting the expression $b_{i|j} = b_{ij} - b_r D^r_{ij}$ by y^i and y^j respectively and then using (4.32)

we get $b_{i|j} y^i y^j = b_{00} - b_r D^r_{00} = \frac{4}{1-4b^2} b_{00}$. (92)

Using (88) and (91), (79) and (80) can be written

$$\frac{1}{\sqrt{1-4b^2}} b_{i0} B^i_\alpha + \sqrt{b^2}\, H_\alpha = 0, \tag{93}$$

$$\frac{1}{\sqrt{1-4b^2}} b_{00} + \sqrt{b^2}\, H_0 = 0, \tag{94}$$

From the equation (94) it is clear that the condition $H_\alpha = 0,$ is equivalent to $b_{00} = 0,$ where b_{ij} is independent of y^i. Since y^i satisfy equation (61), the condition can be written as $b_{ij} y^i y^j = (b_i y^i)(c_j y^j)$ for some

$$2b_{ij} = b_i c_j + b_j c_i\,,$$

$$b_{ij} = \frac{1}{2}(b_i c_j + b_j c_i) \tag{95}$$

Contracting (95) and using the fact that $b_i y^i = 0$, we get $b_{00} = 0\,.$ Multiplying equation (62) by B^i_α and then B^j_β and using equation (60) gives $b_{ij} B^i_\alpha B^j_\beta = 0.$ Similarly we get $b_{ij} B^i_\alpha y^j = 0.$ This further gives $b_{i0} B^i_\alpha y^j = 0.$ Using this in equation (93) gives $H_\alpha = 0.$ Again contracting (95) and then using equation (61) gives $b_{i0} = \frac{b^2 c_0}{2}$. Now using (85) and(86) and using $b_{00} = 0$ and $b_{ij} B^i_\alpha B^j_\beta = 0.$ we get $\lambda^m = 0$, $A^i_j B^j_\beta = 0$ and $B_{ij} B^i_\alpha B^j_\beta = \frac{1}{2\alpha} h_{\alpha\beta}\,.$ Thus using the equation (16),(67),(69),(73) and (82) we get

$$b_r D^r_{ij} \left\{\frac{3\alpha+2\beta+\alpha}{\alpha}\right\}\left\{\frac{\alpha^2-\beta^2}{\alpha^2}\right\} = \frac{c_0 b^2}{4\alpha(1-4b^2)} h_{\alpha\beta}\,. \tag{96}$$

Thus using the relation $b_{ij} = b_{ij} - b_r D^r_{ij}$ and equation (96) equation (78) reduces to

$$\frac{c_0 b^2}{\alpha(1-4b^2)^2} h_{ij} = a_{ij} + 2b_i b_j - y_i y_j + \frac{4b}{\sqrt{((1-4b^2))}}\{a_{ij} + 12 b_i b_j + 4(b_i y_j + b_j y_i)\} N^i B^i_{\alpha\beta} +$$
$$\Gamma^{*i}_{ij}\left\{\frac{3\alpha^3 - 3\alpha\beta^2 + 2\alpha^2\beta - 2\beta^3 + \alpha^3 - \alpha\beta^2}{\alpha^3}\right\} +$$
$$M_\alpha\{a_{ij} + 12 b_i b_j + 4(b_i y_j + b_j y_i)(B^i_{0\beta} + \Gamma^{*i}_{0j}(\frac{3\alpha+2\beta+\alpha}{\alpha}))N^i\}\,. \tag{97}$$

Hence the hyper surface $F^{n-1}(c)$ is umblic.

Theorem 4.4. The necessary and sufficient condition of the Randers change of Square metric for Hypersurface $F^{n-1}(c)$ to be a hyperplane of second kind is given by $b_y b_{ij} = e b_i b_j$. Proof: Since from Lemma (6) $F^{n-1}(c)$ is a hyper plane of second kind if and only if $H_\alpha = 0$ and angular metric tensor is zero. Therefore from (95), implies that $c_0 = c_i(x) y^i = 0$. Thus there exists a function $e(x)$ such that $c_i(x) = e(x) b_i(x)$. Again from (95) the theorem desired. Notice that, from equation (73) and Lemma (7) we deduce that hyper surface F^{n-1} of F is not a hyper plane of third kind.

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