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## RESEARCH ARTICLE

# MINIMAL HYPERSURFACES IN EUCLIDEAN SPACE AND APPLICATIONS

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## ABSTRACT

In this paper we handled hypersurfaces in Euclidean space. We have discussed both the geometrical and analytical characterization of minimal hypersurfaces. The analytical characterization is very much linked to differential equations via the variational principle. We also treated simple cases of differential equations related minimal hypersurfaces.

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## INTRODUCTION

In this section we shall deal with hyper surfaces in a Euclidean space. A smooth parameterized hyper surface in $\mathbb{R}^n$ is a differentiable mapping $r: \Omega \to \mathbb{R}^n$ from an open domain $\Omega$ of $\mathbb{R}^{n-1}$ into the $n$ −dimensional space.

**1.1. Definition:** A parameterized hyper surface is regular if the vectors $r_1(u), \dots, r_{n-1}(u)$ are linearly independent for any $u \in \Omega$. In this case we also say that $r$ is an immersion of the domain $\Omega$ into $\mathbb{R}^n$.

**1.2. Definition:** The affine tangent space of a regular parameterized hypersurface at the point $p = r(u) \in M$ is the hyperplane through $r(u)$ spanned by the direction vectors $r_1(u), \dots, r_{n-1}(u)$. The (linear) tangent space of $M$ at $p$ is the linear space $T_pM$ of the direction vectors of the affine tangent space. The unit normal vector of the hypersurface at the point $(u)$ is defined to be the unit normal vector $N(u)$ of the tangent plane, for which $r_1(u), \dots, r_{n-1}(u), N(u)$ is a positively oriented basis of $\mathbb{R}^n$.

**1.3. Definition:** Let $r: \grave{U} \to \mathbb{R}^n$ be a parameterized hypersurface. A vector field along the hypersurface is a mapping $X: \grave{U} \to \mathbb{R}^n$ from the domain of parameters into $\mathbb{R}^n$. $X$ is a tangential vector field, if $X(u)$ is tangent to the hypersurface at $r(u)$ for all $u \in \grave{U}$.

**1.4. Definition.** Let $r: \grave{U} \to \mathbb{R}^n$ be a parameterized hypersurface, $X: \grave{U} \to \mathbb{R}^n$ be a vector field along it, $u_0 \in \grave{U}$, $v$ a tangent vector of the hypersurface at $r(u_0)$. We define the derivative $\partial_v X$ of the vector field $X$ with respect to the tangent vector $v$ as $\partial_v X = (X \circ u)'(0)$, where $u: [-1,1] \to \grave{U}$ is a curve in the parameter domain such that $u(0) = u_0$, and $(r \circ u)'(0) = v$.

**1.5. Lemma.** The derivative $\partial_v N$ of the unit normal vector field of a hypersurface with respect to a tangent vector $v$ at $p = r(u)$ is tangent to the hypersurface at $r(u)$.

**1.6. Definition:** Let us denote by $M$ the parameterized hypersurface $r: \grave{U} \to \mathbb{R}^n$ and by $T_pM$ the linear space of its tangent vectors at $p = r(u_0)$. The linear map $L_p: T_pM \to T_pM$, $L_p(v) = -\partial_v N$ is called the Weingarten map or shape operator of $M$ at $p$. We define two bilinear forms on each tangent space of the hypersurface.

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**1.7. Definition:** Let $M$ be a parameterized hypersurface $r: Ù \to \mathbb{R}^n$, $u_0 \in Ù$, $T_pM$ the linear space of tangent vectors of $M$ at $p = r(u)$, $L_p: T_pM \to T_pM$ the Weingarten map. The first fundamental form of the hypersurface is the bilinear function $I_p$ on $T_pM$ obtained by restriction of the dot product onto $T_pM$

$I_p(v, w) = \langle v, w\rangle$, for $v, w \in T_pM$.

The second fundamental form of the hypersurface is the bilinear function $II_p$ on $T_pM$ defined by the equality

$II_p(v, w) = \langle L_p v, w\rangle$ for $v, w \in T_pM$.

**1.8. Definition**: For a hypersurface $M$ in $\mathbb{R}^n$ parametrerized by $r$, $r(u_0) = p \in M$, the eigenvalues $k_1(p), \dots, k_{n-1}(p)$ of the Weingarten map $L_p: T_pM \to T_pM$ are called the principal curvatures of $M$ at $p$, the unit eigenvectors of $L_p$ are called principal curvature directions.

**1.9. Definition**: For $M$ a hypersurface, $p \in M$, the determinant $K(p)$ of the Weingarten map $L_p$ is called the Gaussian or Gauss-Kronecker curvature of $M$ at $p$, $H(p) = \frac{tr(L_p)}{n-1}$ is called the mean curvature or Minkowski curvature.

In terms of the principal curvatures, they are

$K = k_1 k_2 \dots k_{n-1}$, $\qquad H = \frac{1}{n-1}(k_1 + \cdots + k_{n-1})$.

**1.10. Definition:** A hypersurface $M$ in $\mathbb{R}^n$ for which mean curvature is identically zero is called a minimal hypersurface.

**1.11. Theorem:** (W. Kühnel, 2015**)** All points of a connected surface element of class $C^2$are umbilics if and only if the surface is contained in either a plane or a sphere.

An umbilic is defined as a point in which the Weingarten map is a multiple of the identity.

**1.12. Definition:** The $n$-dimensional Monge coordinates are given by

$f(u_1, \dots, u_n) = (u_1, \dots, u_n, h(u_1, \dots, u_n))$

with the unit normal $v(u_1, \dots, u_n) = \frac{1}{\sqrt{1+h_{u_1}^2+\cdots+h_{u_n}^2}}\left(-h_{u_1}, \dots, -h_{u_n}, 1\right)$.

**1.13. Definition:** (Y. Aminov, 2001) For the hypersurface the volume element has the form

$dV = \left|\left[r_{u_1}, \dots, r_{u_n}\right]\right| du^1 \dots du^n$.

**1.14. Theorem:** (W. Kühnel, 2015) A connected surface element of the class $C^2$ consists only of umbilics if and only if it is contained in a hyperplane or a hypershere. It is said to be totally umbilical.

**2. Generalization of minimal surface equation**

Let $Ó^k \subset (M^n, g)$ be a $k$-dimensional submanifold of an $n$-dimensional Riemannian manifold $(M^n, g)$, where $\nabla$ is the corresponding Riemannian connection. Let $g|_Ó$ be the induced metric. Given tangent vectors $X$, $Y$ of $\Sigma$, the second fundamental form, which is a vector valued symmetric 2-tensor on $\Sigma$, is defined as:

$$\vec{B}(X, Y) = (\nabla_X Y)^{\perp}. \tag{2.1}$$

2.1. Definition: The mean curvature of $\Sigma$ is defined as:

$\vec{H} = Tr_g \vec{B} = \sum_{i,j=1}^{k} g^{ij} \vec{B}(e_i, e_j)$, where $\{e_i\}_{i=1}^{k}$ is a tangent basis of $\Sigma$.

2.2. Definition: $\Sigma$ is called minimal if $\vec{H} = 0$.

Consider a hypersurface $Ó_f^{n-1} \subset \mathbb{R}^n$ as a graph

$Ó_f = \{(u, f(u)): u = (u_1, \dots, u_{n-1}) \in Ù\}$

of a function $f$, where $Ù \subset \mathbb{R}^n$. Denote

$G(u_1, \dots, u_{n-1}) = (u_1, \dots, u_{n-1}, f(u))$.

Then the induced metric is given by $g_{ij} = G_{u_i} \cdot G_{u_j} = \text{ä}_{ij} + f_i f_j$, where $f_i = \frac{\partial f}{\partial u_i}$. The matrix $(\text{ä}_{ij} + f_i f_j)$ has $n-2$ multiple eigenvalues 1 with eigenspace $(\nabla f)^{\perp}$ and a single eigenvalue $1 + |\nabla f|^2$ with eigenvector $\nabla f$.

If we think $f$ as a function defined on $\mathbb{R}^n$, then the graph $\text{Ó}_f$ is the level set given by $u_n - f(u) = 0$. So the unit normal of $\text{Ó}_f$ is given by

$N = \frac{(-\nabla f, 1)}{\sqrt{1+|\nabla f|^2}}$.

Hence the inverse matrix for $g_{ij}$ is given by $g^{ij} = \text{ä}_{ij} - \text{í}_i \text{í}_j$, where $\text{í}_i = \frac{f_i}{\sqrt{1+|\nabla f|^2}}$ for $i = 1, \dots, n-1$.

Now the volume form of $\text{Ó}_f$ is given by $dV = \sqrt{\det g}\ du \ = \sqrt{1 + |\nabla f|^2}\, du$. So the volume of $\text{Ó}_f$ is:

$|\text{Ó}_f| = \int_{\text{Ù}} \sqrt{1 + |\nabla f|^2}\, du.$

Now let us calculate the Euler-Lagrange equation for $|\text{Ó}_f|$. For any $\text{ç} \in C_c^{\infty}(\text{Ù})$, suppose $f$ is a critical point of $|\text{Ó}_f|$, then

$$\frac{d}{dt}\Big|_{t=0} = |\text{Ó}_{f+t\text{ç}}| = \int_{\text{Ù}} \frac{\nabla f \cdot \nabla \text{ç}}{\sqrt{1+|\nabla f|^2}} du = -\int_{\text{Ù}} \frac{\partial}{\partial u_i}\left(\frac{f_i}{\sqrt{1+|\nabla f|^2}}\right) \text{ç} du = 0 \quad (2.2)$$

So the divergence forms of the minimal surface equation:

$$(i): \quad \sum_{i=1}^{n-1} \frac{\partial}{\partial u_i}\left(\frac{f_i}{\sqrt{1+|\nabla f|^2}}\right) = 0 \quad (2.3)$$

Expanding the above equation, we get:

$$\frac{f_{u_i u_i}}{\sqrt{1+|\nabla f|^2}} - \frac{f_{\_i} f_{u_j} f_{u_i u_j}}{(1+|\nabla f|^2)^{3/2}} = 0,$$

which can be rewritten as:

$$\frac{1}{\sqrt{1+|\nabla f|^2}}\left(\text{ä}_{ij} - \frac{f_i}{\sqrt{1+|\nabla f|^2}} \frac{f_i}{\sqrt{1+|\nabla f|^2}}\right) f_{u_i u_j} = 0, \quad (2.4)$$

So we get the non divergence form of minimal surface equation:

$$(ii): \quad \frac{1}{\sqrt{1+|\nabla f|^2}} \sum_{i,j=1}^{n-1} g^{ij} f_{u_i u_j} = 0 \ , \quad (2.5)$$

where $g^{ij} = \text{ä}_{ij} - \text{í}_i \text{í}_j$ is inverse matrix for the induced metric. Now let us calculate the mean curvature of $\text{Ó}_f$ from definition. Firstly,

$$\vec{B}\left(\partial_{u_i}, \partial_{u_j}\right) = \left(G_{u_i u_j}\right)^{\perp} = \underbrace{\left(G_{u_i u_j} \cdot N\right)}_{h_{ij}} N$$

It is easy to see that

$$G_{u_i u_j} = \left(0, \dots, 0, f_{u_i u_j}\right),$$

So $h_{ij} = \frac{f_{u_i u_j}}{\sqrt{1+|\nabla f|^2}}$. So it is easy to see that equation (2.5) is equivalent with equation (2.4).

From $H = 0$ it follows that $g^{ij} f_{u_i u_j} = 0$. Thus we obtain the minimal hypersurface equation

$$(1 + \sum_{i=1}^{n-1} f_i^2) f_{ij} - f_i f_j f_{ij} = 0 \quad (2.6)$$

which is equivalent to equation (2.3).

When $n = 3$ in equation (2.6) reduce to:

$$(1+f_{u_2}^2)f_{u_1u_1} - 2f_{u_1}f_{u_2}f_{u_1u_2} + (1+f_{u_1}^2)f_{u_2u_2} = 0 \tag{2.7}$$

Consider a hypersurface $M$ given by $z = f(u_1, u_2, u_3)$ and parametrize by it $X(u_1, u_2, u_3) = (u_1, u_2, u_3, f(u_1, u_2, u_3))$ .Then, it is easy to compute the coefficients of the first and second fundamental form of a function as

$$X_{u_1} = (1, 0, 0, f_{u_1}) \tag{2.8}$$
$$X_{u_2} = (0, 1, 0, f_{u_2}) \tag{2.9}$$
$$X_{u_3} = (0, 0, 1, f_{u_3}) \tag{2.10}$$
$$N = \frac{X_{u_1} \wedge X_{u_2} \wedge X_{u_3}}{|X_{u_1} \wedge X_{u_2} \wedge X_{u_3}|} = \frac{(f_{u_1}, f_{u_2}, f_{u_3}, -1)}{\sqrt{1+f_{u_1}^2+f_{u_2}^2+f_{u_3}^2}} \tag{2.11}$$

The matrix of the first fundamental form is

$$g_{ij} = \begin{pmatrix} 1+f_{u_1}^2 & f_{u_1}f_{u_2} & f_{u_1}f_{u_3} \\ f_{u_2}f_{u_1} & 1+f_{u_2}^2 & f_{u_2}f_{u_3} \\ f_{u_3}f_{u_1} & f_{u_3}f_{u_2} & 1+f_{u_3}^2 \end{pmatrix} \tag{2.12}$$

The inverse matrix for $g_{ij}$ is given by

$$g^{ij} = \frac{1}{\sqrt{1+f_{u_1}^2+f_{u_2}^2+f_{u_3}^2}} \begin{pmatrix} 1+f_{u_2}^2+f_{u_3}^2 & -f_{u_1}f_{u_2} & -f_{u_1}f_{u_3} \\ -f_{u_2}f_{u_1} & 1+f_{u_1}^2+f_{u_3}^2 & -f_{u_2}f_{u_3} \\ -f_{u_3}f_{u_1} & -f_{u_3}f_{u_1} & 1+f_{u_1}^2+f_{u_2}^2 \end{pmatrix} \tag{2.13}$$

$$X_{u_1u_1} = (0, 0, 0, f_{u_1u_1}) \tag{2.14}$$
$$X_{u_2u_2} = (0, 0, 0, f_{u_2u_2}) \tag{2.15}$$
$$X_{u_3u_3} = (0, 0, 0, f_{u_3u_3}) \tag{2.16}$$
$$X_{u_1u_1} \cdot N = \frac{1}{\sqrt{1+f_{u_1}^2+f_{u_2}^2+f_{u_3}^2}}(-f_{u_1u_1}) \tag{2.17}$$
$$X_{u_2u_2} \cdot N = \frac{1}{\sqrt{1+f_{u_1}^2+f_{u_2}^2+f_{u_3}^2}}(-f_{u_2u_2}) \tag{2.18}$$
$$X_{u_3u_3} \cdot N = \frac{1}{\sqrt{1+f_{u_1}^2+f_{u_2}^2+f_{u_3}^2}}(-f_{u_3-_3}) \tag{2.19}$$

Let us find $g = \det\|g_{ij}\|$. Hence

$$g = 1 + f_{u_1}^2 + f_{u_2}^2 + f_{u_3}^2 \tag{2.20}$$

Now we ready to find the formula for the mean curvature $H$. Let $B = (b_{ij})$ the matrix representation of the second fundamental form, i.e.,

$$B = \frac{1}{\sqrt{g}} \begin{pmatrix} -f_{u_1u_1} & -f_{u_1u_2} & -f_{u_1u_3} \\ -f_{u_2u_1} & -f_{u_2u_2} & -f_{u_2u_3} \\ -f_{u_3u_1} & -f_{u_3u_2} & -f_{u_3u_3} \end{pmatrix} \tag{2.21}$$

From (2.13) and (2.21) we find

$$L_P = Bg^{ij} = \frac{1}{g} \begin{pmatrix} o & p & q \\ r & s & t \\ u & v & w \end{pmatrix} \tag{2.22}$$

Where

$$o = -f_{u_1u_1}(1+f_{u_2}^2+f_{u_3}^2) + f_{u_1u_2}(f_{u_2}f_{u_1}) + f_{u_1u_3}(f_{u_3}f_{u_1}),$$
$$p = f_{u_1u_1}(f_{u_1}f_{u_2}) - f_{u_1u_2}(1+f_{u_1}^2+f_{u_3}^2) + f_{u_1u_3}(f_{u_3}f_{u_2}),$$
$$q = f_{u_1u_1}(f_{u_1}f_{u_3}) + f_{u_1u_2}(f_{u_2}f_{u_3}) - f_{u_1u_3}(1+f_{u_1}^2+f_{u_2}^2),$$
$$r = -f_{u_2u_1}(1+f_{u_2}^2+f_{u_3}^2) + f_{u_2u_2}(f_{u_2}f_{u_1}) + f_{u_2u_3}(f_{u_3}f_{u_1}),$$
$$s = f_{u_2u_1}(f_{u_1}f_{u_2}) - f_{u_2u_2}(1+f_{u_1}^2+f_{u_3}^2) + f_{u_2u_3}(f_{u_3}f_{u_2}),$$

$t = f_{u_2u_1}(f_{u_1}f_{u_3}) + f_{u_2u_2}(f_{u_2}f_{u_3}) - f_{u_2u_3}(1 + f_{u_1}^2 + f_{u_2}^2)$,
$u = -f_{u_3u_1}(1 + f_{u_2}^2 + f_{u_3}^2) + f_{u_3u_2}(f_{u_2}f_{u_1}) + f_{u_3u_3}(f_{u_3}f_{u_1})$,
$v = f_{u_3u_1}(f_{u_1}f_{u_2}) - f_{u_3u_2}(1 + f_{u_1}^2 + f_{u_3}^2) + f_{u_3u_3}(f_{u_3}f_{u_2})$,
$w = f_{u_3u_1}(f_{u_1}f_{u_3}) + f_{u_3u_2}(f_{u_2}f_{u_3}) - f_{u_3u_3}(1 + f_{u_1}^2 + f_{u_2}^2)$.

Thus,

$$H = \frac{1}{3} trace\ L_p$$
$$= \frac{f_{u_1u_1}(1 + f_{u_2}^2 + f_{u_3}^2) + f_{u_2u_2}(1 + f_{u_1}^2 + f_{u_3}^2) + f_{u_3u_3}(1 + f_{u_1}^2 + f_{u_2}^2) - 2f_{u_1u_2}(f_{u_1}f_{u_2}) - 2f_{u_1u_3}(f_{u_1}f_{u_3}) - 2f_{u_2u_3}(f_{u_2}f_{u_3})}{3(1 + f_{u_1}^2 + f_{u_2}^2 + f_{u_3}^2)^{3/2}}$$

Therefore $M$ is a minimal hypersurface if and only if $H = 0$ . We get the following partial differential equation:

$$f_{u_1u_1}(1 + f_{u_2}^2 + f_{u_3}^2) + f_{u_2u_2}(1 + f_{u_1}^2 + f_{u_3}^2) + f_{u_3u_3}(1 + f_{u_1}^2 + f_{u_2}^2) - 2f_{u_1u_2}(f_{u_1}f_{u_2}) - 2f_{u_1u_3}(f_{u_1}f_{u_3}) - 2f_{u_2u_3}(f_{u_2}f_{u_3}) = 0.$$

**2.3. Theorem:** (J. Moser, 1961) Let $z = f(u^1, \dots, u^n)$ defined an entire minimal graph in $\mathbb{R}^{n+1}$. If $|\nabla f| \le$ â $< \infty$, then $f$ is a linear function and its graph a hyperplane.

**Bernstein's Problem:** Let $u_1, \dots, u_n,\ z$ be standard coordinates in $\mathbb{E}^{n+1}$, consider minimal hypersurface which can be represented by an equation of the form

$z = f(u_1, \dots, u_n)$

for all $u_i$, that is, which have a one-to-one projection onto a hyperplane. The answer to Bernstein's problem is known to be affirmative in the following cases:

- $n = 2$ [Bernstein 1914];
- $n = 3$ [de Giorgi 1965];
- $n = 4$ [Almgren 1966], and
- $n = 5,6,7$ [Simons 1968].

and the answer is negative for $n \ge 8$ [Bombieri- de Giorgi- Giusti 1969]. In general, when a bounded domain $D$ in $\mathbb{E}^n$ and continuous function $\varphi$ on its boundary $\partial D$ are given, the problem of finding a minimal hypersurface $M$ defined by the graph of a real valued-function of on $\overline{D}$, the closure of $D$, with $f|_{\partial D} =$ ö gives rise to a typical Dirichlet problem. The questions are those of existence, uniqueness, and regularity of solutions. These problems were studied by T. Radó for $n = 2$ and later by L. Bers, R. Finn, H. Jenkins, J. Serrin, R. Osserman, and others.

**First Variation of the Area:** We know that a minimal surface in $\mathbb{R}^3$ as surface whose mean curvature vanishes. A more intuitive meaning of minimal surface is the surface of least area among a family of surfaces having the same boundary. In order to show how these two definitions coincide, we define the normal variation of a surface $M$ in $\mathbb{R}^3$ to be a family of surfaces $t \mapsto M(t)$ representing how $M$ changes when pulled in a normal direction. Let $(t)$ denote the area of $(t)$. We show that the mean curvature of $M$ vanishes if and only if the first derivative of $t \mapsto A(t)$ vanishes at $M$. 4.1. Definition. Let $X: U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular parametrized surface. Choose a bounded region $D \subset U$ and a differentiable function $h: \overline{D} \to \mathbb{R}$, where $\overline{D}$ is the union of the region $D$ with its boundary $\partial D$. Let $N$ denote a unit normal vector field such that $(u,$ v$)$ is perpendicular to X$(u, v)$ for all $u, v \in U$. Then the normal variation of $X(\overline{D})$, determined by the function $h$, is a map

ö: $\overline{D} \times (-\epsilon, \epsilon) \to \mathbb{R}^3$

given by ö$(u, v, t) = X(u, v) + t \cdot h(u, v) \cdot N(u, v)$, $(u, v) \in \overline{D}$, $t \in (-\epsilon, \epsilon)$, where $\epsilon \in \mathbb{R}$, $\epsilon > 0$. We now employ the notion of a normal variation to show that a minimal surface is a critical point of the area functional. For each fixed $t \in (-\epsilon, \epsilon)$, the map

$X^t: D \to \mathbb{R}^3$
$X^t(u, v) =$ ö$(u, v, t)$

is a parametrized surface with
$\frac{\partial X^t}{\partial u} = X_u + thN_u + th_uN,$

$\frac{\partial x^t}{\partial v} = X_v + thN_v + th_v N.$

Thus, if we denote by $E^t$, $F^t$, $G^t$ the coefficients of the first fundamental form of $X^t$, we Obtain

$E^t = \frac{\partial x^t}{\partial u} \cdot \frac{\partial x^t}{\partial u} = E + th(\langle X_u, N_u \rangle + \langle X_u, N_u \rangle) + t^2h^2 \langle N_u, N_u \rangle + t^2 h_u h_u,$

$F^t = \frac{\partial x^t}{\partial u} \cdot \frac{\partial x^t}{\partial v} = F + th(\langle X_u, N_v \rangle + \langle X_v, N_u \rangle) + t^2h^2 \langle N_u, N_v \rangle + t^2 h_u h_v,$

$G^t = \frac{\partial x^t}{\partial v} \cdot \frac{\partial x^t}{\partial v} = G + th(\langle X_v, N_v \rangle + \langle X_v, N_v \rangle) + t^2h^2 \langle N_v, N_v \rangle + t^2 h_v h_v.$

By using the fact

$\langle X_u, N_u \rangle = -e, \quad \langle X_u, N_v \rangle + \langle X_v, N_u \rangle = -2f, \quad \langle X_v, X_v \rangle = -g$

and that the mean curvature $H$ is

$H = \frac{1}{2}\left\{\frac{Eg-2fF+Ge}{EG-F^2}\right\},$

we obtain

$E^tG^t - (F^t)^2 = EG - F^2 - 2th(Eg - 2Ff + Ge) + R$

$= (EG - F^2)(1 - 4thH) + R,$

where $\lim_{t\to 0}\left({R}/{t}\right) = 0.$

It follows that if $\epsilon$ is sufficiently small, $X^t$ is a regular parametrized surface. Furthermore, the area $(t)$ of $X^t(\bar{D})$ is

$$A(t) = \int_{\bar{D}} \sqrt{E^tG^t - (F^t)^2}\, dudv$$
$$= \int_{\bar{D}} \sqrt{1 - 4thH + \bar{R}} \quad \sqrt{EG - F^2} dudv,$$

where $\bar{R} = {R}/{(EG - F^2)^2}$. It follows that if $\epsilon$ is small, $A$ is a differentiable function and its derivative at $t = 0$ is

$$A'(0) = -2\int_{\bar{D}} hH\sqrt{EG - F^2}\, dudv$$

These results provide us with necessary tools to prove the following.

**4.2. Proposition:** Let $X: U \to \mathbb{R}^3$ be a regular parametrized surface and let $D \subset U$ be a bounded domain in $U$. Then $X$ is minimal if and only if $A'(0) = 0$ for all such $D$ and all normal variation of $X(\bar{D})$.

**4.3. Definition:** A smooth variation of a regular parametrized hypersurface $r$: Ù $\to \mathbb{R}^n$ is a smooth map $R$: Ù $\times$ (−å, å) $\to \mathbb{R}^n$, such that for all $t \in$ (−å, å), the map $r_t$: Ù $\to \mathbb{R}^n$ defined by $r_t(u) = R(u,t)$ is a regular parametrized hypersurface and $r_0 = r$. A variation is compactly supported if there is a compact set $K \subset$ Ù, such that $R(u,t) = r(u)$ for all $u \notin K$ and $t \in$ (−å, å). For each point $r(u)$ of the hypersurface, the variation $R$ defines a parametrized curve $R(u,.)$: (−å, å) $\to \mathbb{R}^n$ which describes a motion of that point. The image of the parametrized hypersurface $r_t$ consists of those points at which the points of the original hypersurface $r$ arrive after moving on for time $t$.

**The First Variation Formula of Volume:** A minimal submanifold is defined to be one with vanishing mean curvature. This definition seems to have no relation with "minimal" terminology. In fact, Lagrange found minimal surfaces in his investigation of the calculus of variations. Now, we generalize Lagrange's study to more general setting. First of all we need the following algebraic result.

**5.1. Lemma:** Let $A(t) = \left(a_{ij}(t)\right)$, $|t| < \epsilon$ be a smooth family of $n \times n$ matrices satisfying $A(0) = \mathrm{I}$(the identity matrix). Then

$\frac{d}{dt}\det A(t)\Big|_{t=0} = traceA'(0).$

Now, we can derive the first variational formula.

**5.2. Theorem:** Let $M$ be a compact Riemannian manifold, $f: M \to \widetilde{M}$ an isometric immersion with mean curvature vector $H$. Let $f_t$, $|t| < \epsilon$, $f_0 = f$, be a smooth family of immersion satisfying $f_t|_{\partial M} = f|_{\partial M}$. Denote $X = \frac{\partial f_t}{\partial t}\Big|_{t=0}$ to be the variational vector field along $f$. Then

$\frac{d}{dt}\, vol(f_t M)\Big|_{t=0} = -\int_M \langle nH, X\rangle dvol.$

A submanifold with vanishing first variation for all compactly supported vector fields $X$ is said to be stationary, and we see from the first variation formula that a smooth submanifold is stationary if and only if it is minimal.

**The second variation formula of Volume:** We know from the first variational formula that minimal immersion $f: M \to \widetilde{M}$ is a critical point on the immersion space from $M$ into $\widetilde{M}$. It is natural to ask if $f$ is a local minimum of the volume functional, namely, for any smooth variations $f_t: M \to \widetilde{M}$, at $t > 0$ small enough whether

$vol(f) \leq vol(f_t)$

holds true. To answer the problem we need to derive the second variational formula. First of all let us consider the relevant geometric invariants.

(1) For cross-sections on a vector bundle we can define the trace-Laplace operator $\nabla^2$. For the immersion $f: M \to \widetilde{M}$ we have normal bundle $NM$, where there define an induced connection $\nabla = (\widetilde{\nabla})^N$ on normal bundle. Hence,
(2)
$\nabla^2$: $\tilde{A}(NM) \to \tilde{A}(NM)$.

We assume that $M$ has boundary $\partial M \neq \emptyset$, $-\nabla^2$ is self adjoint, semi-positive operator on $\mathcal{N}_0 = \{$í $\in \tilde{A}(NM);$ í$|_{\partial M} \equiv 0\}$.
It is also an elliptic operator.

(3) The second invariant is defined by curvature of the ambient manifold. Let $\tilde{R}$ be the Riemannian curvature tensor on $\widetilde{M}$. Define $\tilde{R} \in Hom(N_pM,\ N_pM)$ as follows:

$\tilde{R}$(í) $= \{\tilde{R}_{\text{í}e_i}(e_i)\}^N$

Where í $\in N_pM$ and $\{e_i\}$ is a local orthonormal frame field near $p \in M$. It is a symmetric operator owing to the properties of the curvature tensor.

In fact ì, í $\in NM$,

$\langle\tilde{R}(\text{í}), \text{ì}\rangle = \langle\tilde{R}_{\text{í}e_i}(e_i), \text{ì}\rangle = \langle\tilde{R}_{e_i\text{ì}}(\text{í}), e_i\rangle = \langle\tilde{R}_{\text{ì}e_i}(e_i), \text{í}\rangle = \langle\tilde{R}(\text{ì}), \text{í}\rangle$

**Remark.** If the codimension of $M$ in $\widetilde{M}$ is one, we call equation 1.6.

$\tilde{R}(\text{í}) = \left(\tilde{R}_{\text{í}e_i}(e_i)\right)^N = -\widetilde{Ric}(\text{í}, \text{í})\text{í}.$

(4) The last invariant involves the second fundamental form $B$ of $M$ in $\widetilde{M}$. Recall that $B \in \tilde{A}(Hom(S^2TM, NM))$. Its adjoint operator is $A = B^T \in Hom(NM, S^2TM)$. We define Ƀ $\in \tilde{A}(Hom(NM, NM))$ by

Ƀ $= B \circ B^T$

(6. 2)

By definition it follows that

$\langle$Ƀ$(\text{í}), \text{ì}\rangle = \langle B \circ B^T(\text{í}), \text{ì}\rangle = \langle B^T(\text{í}), B^T(\text{ì})\rangle = \langle B^T(\text{í}), e_i \otimes e_j\rangle\langle B^T(\text{ì}), e_i \otimes e_j\rangle \quad = \langle B_{e_ie_j}, \text{í}\rangle\,\langle B_{e_ie_j}, \text{ì}\rangle.$

Hence, Ƀ is symmetric and semi positive.

**6.1. Theorem:** (Y. Xin, 2003) Let $f: M \to \widetilde{M}$ be a compact minimal immersion, í $\in \mathcal{N}_0$ be a normal vector field which vanishes on $\partial M$. Assume that $f_t: M \to \widetilde{M}$ is a smooth one parameter family of immersions, such that for $|t| < \epsilon$

$$\begin{cases} f_0 = f \\ \left.\frac{\partial f_t}{\partial t}\right|_{t=0} = \text{í} \\ f_t|_{\partial M} = f|_{\partial M} \end{cases}$$

for each $t$. Then

$$\left.\frac{d^2}{dt^2} vol(f_t M)\right|_{t=0} = \int_M \langle -\nabla^2 \text{í} + \tilde{R}(\text{í}) - \text{Ɓ}(\text{í}), \text{í}\rangle * 1 \qquad (6.\ 3)$$

The second variational formula (6. 3) indicates that it is useful to study the elliptic differential operator of second order defined on $\mathcal{N}_0$

$$S = -\nabla^2 + \tilde{R} - \text{Ɓ}.$$

This is so called Jacobi operator. We thus can define a symmetric bilinear form on $\mathcal{N}_0$

$$\mathrm{I}(\text{ì}, \text{í}) = \int_M \langle S(\text{ì}), \text{í}\rangle * 1.$$

**6.2. Definition:** Let $M \to \widetilde{M}$ be a minimal immersion. If for any $\text{ì} \in \mathcal{N}_0$, $\mathrm{I}(\text{ì}, \text{ì}) > 0 (\geq 0)$, then $M$ is called stable (weakly stable) minimal submaifold. We now study the codimension one case, moreover the normal bundle is assumed to be trivial. Thus we have the unit normal vector field $\nu$. If $\text{ì} \in \mathcal{N}_0$, then $\text{ì} = \text{öí}$, where $\text{ö} \in C^\infty(M)$ and $\text{ö}|_{\partial M} = 0$. Hence

$$\tilde{R}(\text{ì}) = \sum \left(\tilde{R}_{\text{ì}e_i} e_i\right)^N = \text{ö}\left(\sum \tilde{R}_{\text{í}e_i} e_i\right)^N = \text{ö}\left(\tilde{R}_{\text{í}e_i} e_i + \tilde{R}_{\text{íí}}\text{í}\right)^N = -\text{ö}\left(\widetilde{Ric}(\text{í})\right)^N,$$

$$\text{Ɓ}(\text{ì}) = \langle \text{Ɓ}(\text{ì}), \text{í}\rangle \text{í} = \langle B_{e_i e_j}, \text{ì}\rangle \langle B_{e_i e_j}, \text{í}\rangle\, \text{í} = \text{ö}\, \langle B_{e_i e_j}, \text{í}\rangle \langle B_{e_i e_j}, \text{í}\rangle\, \text{í} = |B|^2 \text{ì},$$

$$\nabla_{e_i} \text{ì} = \nabla_{e_i} \text{öí} = \left(\nabla_{e_i} \text{ö}\right)\text{í}.$$

The second variational formula the becomes

$$\mathrm{I}(\text{ì}, \text{ì}) = \int_M \left(|\nabla \text{ö}|^2 - (\langle \widetilde{Ric}\text{í}, \text{í}\rangle + |B|^2 \text{ö}^2)\right) * 1 \qquad (6.\ 4)$$

When the ambient manifold is Euclidean space $\mathbb{R}^{n+1}$,

$$\mathrm{I}(\text{ì}, \text{ì}) = \int_M (|\nabla \varphi|^2 - |B|^2 \text{ö}^2) * 1 \qquad (6.\ 5)$$

6.3. Proposition. (Y. Xin, 2003) The minimal graph $M$ in $\mathbb{R}^{n+1}$, which defined by $x^{n+1} = f(x^1, \ldots, x^n)$, is weakly stable. In fact, the minimal graphs are area minimizing. Let $M_D \subset M$ be any bounded domain, $\widetilde{M}$ be any hypersurface with $\partial M = \partial M_D$, $\sum \subset \mathbb{R}^{n+1}$ be a domain bounded by $\widetilde{M}$ and $M$. Consider a vector field $X$ on $\sum$ which can be obtained by parallel translating the unit normal vector $\nu$ to $M_D$ along the $x^{n+1}$ axis. Then by the minimal surface equation (2. 3) we have

$$div\, X = 0$$

Hence, by Green's theorem

$$0 = \int_\Sigma div\, X dV_\Sigma = \int_{\partial\Sigma} \langle X, n\rangle dV_{\partial\Sigma},$$

where $n$ is a unit normal vector field to $\partial\sum = M_D - \widetilde{M}$. We then have

$$Vol(M_D) = \int_{M_D} \langle X, n\rangle dV_M = \int_{\widetilde{M}} \langle X, n\rangle dV_{\widetilde{M}} \leq \int_{\widetilde{M}} |X||n| dV_{\widetilde{M}} = Vol(\widetilde{M})\,. \qquad (6.\ 6)$$

Let us explain the relations between minimal graphs, area minimizing hypersurfaces and stable hypersurfaces. A minimal hypersurface ($H = 0$) is a critical point of the volume with respect to deformations with compact support. A minimal hypersurface is called stable if the second variation of the volume is nonnegative for all compactly supported deformations. An area minimizing hypersurface is a minimum for the volume and of course an area minimizing hypersurface is stable. Furthermore a minimal graph $M$ over a domain of $\mathbb{R}^n$ is stable.

**Minimal Surfaces and Laplace's Equation:** At first the geometric context. Take a closed wire frame, dip it in soap solution and pull it out. What is the shape of the soap film? One mathematical formulation of this problem is the following. Let us suppose the surface can be described as a graph over an open subset $U$ of $\mathbb{R}^2$. Any smooth function $f: U \to \mathbb{R}$, yields a surface $z = f(u, v)$. The area of this surface is

$$I(f) = \int_U \sqrt{1+|\nabla f|^2}\, dudv \tag{7. 1}$$

If the shape of the wire frame is known (say given by a function $g: \partial U \to \mathbb{R}^3$), we require that $f$ satisfy the boundary condition $f = g$ for $x \in \partial U$. Thus, the problem is to find a function $f$ that minimizes $I(f)$ subject to the boundary condition. In calculus you learn that the gradient of a function must vanish at an extremum. Our problem is similar. We wish to minimize a functional, thus we must find the analogue of its gradient. The way to this is to assume that one has a solution to the problem, and then say that the first order changes must be zero. Suppose we have found a solution $f$, then if we consider a variation $\varphi$ that is smooth and ö $= 0$ on $\partial U$, we expect that

$$\left.\frac{dI(f+\text{åö})}{d\text{å}}\right|_{\text{å}=0} = 0 \tag{7. 2}$$

We substitute in (7.1) and integrate by parts to find

$$0 = \int \frac{\nabla\text{ö}\cdot\nabla f}{\sqrt{1+|\nabla f|^2}}\, dudv = -\int \text{ö}div\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) dudv \tag{7. 3}$$

There is a wonderful trick at this point. Since $\varphi$ is arbitrary, we can in fact deduce that $f$ must satisfy the minimal surface equation:

$$\operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = 0, \quad x \in U \tag{7. 4}$$

Observe that there is no real need to suppose that the surface is two-dimensional, and the same equation would result for higher dimensional minimal surfaces. The geometric quantity on the left hand side is $n$ times the mean curvature; thus minimal surfaces must have zero mean curvature. The minimal surface equation is nonlinear, and unfortunately rather hard to analyze. A simpler version of the equation is obtained by linearization: we assume that $|\nabla f|^2 \ll 1$ and neglect it in the denominator. Thus, we are led to Laplace's equation

$$\operatorname{div}(\nabla f) = 0 \tag{7. 5}$$

The combination of derivatives div $\nabla = \sum_{i=1}^{n} \partial_{u_i}^2$ arises so often that it is denoting $\Delta$. The combination of the PDE and boundary condition on $f$ is called the Dirichlet problem

$$\Delta f = 0, \quad x \in U, \quad f = g, \quad x \in \partial U \tag{7. 6}$$

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