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RESEARCH ARTICLE

AN STOCHASTIC SIRS MODEL WITH NONLINEAR INCIDENCE RATE AND TRANSFER FROM INFECTIOUS TO SUSCEPTIBLE

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ABSTRACT

In this paper, we studied that asymptotic behaviors of an stochastic SIRS model with nonlinear incidence rate and transfer from infectious to susceptible. First of all, we given the global existence and positivity of the solutions. Moreover, we found that stochastic perturbation in the environment can lead the disease to extinction under certain conditions. Furthermore, we established that sufficient conditions for the existence of a stationary probability measure of the model. Finally, we made a brief conclusion.

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INTRODUCTION

Infectious diseases are big threats not only the health of individuals, but also societies, economical and political systems. Therefore, it is very important to understand the mechanisms of the spread of epidemics or the factors that contribute to their occurrence in order to fight them or at least to bring them under control. Mathematical modeling describing the population dynamics of infectious disease areused extensively. One of the simplest epidemic model is the Kermack-Mckendrickmodel (Kermack *et al.,* 1927), which is separated the population into three compartments of susceptible,infective and recovered individuals according to their status relative to the disease, with numbers at time t denoted by $S(t)$, $I(t)$ and $R(t)$ respectively. Since then,many scholars have made studies on them (Lahrouz *et al.,* 2011; Buonomo and Rionero, 2010; Korobeinikov, 2006; Beretta, 1995; Lu *et al.,* 2002). It is well known that the incidence rate is an important factor in the transmission of infectious disease. In many previous epidemic models, the bilinear incidence rate βSI and standard incidence rate $\frac{\beta SI}{N}$ was frequently used (Kermack, 1927; Hethcote and Herbert, 2000; Mena-Lorca and Hethcote, 1992; Liu *et al.,* 2017; Li *et al.,* 2013). In Li *et al.* propose the following SIRS epidemic model with a nonlinear incidence $Sf(I)$ and transfer from the infected class to the susceptible class,

$$\begin{cases} \frac{dS}{dt} = \lambda - \mu S - Sf(I) + \gamma_1 I + \delta R \\ \frac{dI}{dt} = Sf(I) - (\mu + \gamma_1 + \gamma_2 + \alpha) I \\ \frac{dR}{dt} = \gamma_2 I - (\mu + \delta) R \end{cases} \tag{1.1}$$

with the initial conditions

$$S(0) = S_0 \geq 0, I(0) = I_0 \geq 0, R(0) = R_0 \geq 0, \tag{1.2}$$

Where the parameters are:

- λ : recruitment rate of susceptible individuals.
- μ : natural death rate.
- γ_1 : transfer rate from the infected class to the susceptible class.
- γ_2 : transfer rate from the infected class to the recovered class.
- α : disease-induced death rate.
- δ : immunity loss rate.

λ and μ are assumed to be positive, while the parameters α , γ_1, γ_2 and δ are non-negative.

In addition, f is a real locally Lipschitz function on $\mathbb{R}_+ = [0,+\infty)$ satisfying

$(I): f(0)=0$ and $f(I)>0$ for all $I \ge 0$;

$(II): \frac{f(I)}{I}$ is continuous and monotonously nonincreasing for $I>0$ and $\lim_{I\to 0^+} \frac{f(I)}{I} = \beta$ exists, denoted by β with $\beta > 0$.

It is easy to see from (II) that

$$f(I) \le \beta I, \text{ for all } I \in R_+. \tag{1.3}$$

According to the basic reproduction number for system (1.1) is $R_0 = \frac{\lambda\beta}{\mu(\mu+\gamma_1+\gamma_2+\alpha)}$. when $R_0<1$, the disease-free equilibrium $E_0 = (\frac{\lambda}{\mu}, 0, 0)$ is asymptotically stable. When $R_0>1$, then E_0 is unstable and there is a globally asymptotically stable endemic equilibrium $E^* = (S^*, I^*, R^*)$.

In fact, the spread of disease is inevitably disturbed by the environmental noise, which can provide an additional degree of realism in compared to their deterministic counterparts. Therefore, it is very necessary to study how noise affects the epidemic model. May (2007) has revealed that due to environmental fluctuation, the birth rate, death rate, transmission coefficient and other parameters involved with the system exhibit random fluctuations to a greater or lesser extent. Consequently, many researchers introduced stochastic perturbations into deterministic models and obtained some excellent results (see (Zhang *et al.,* 2014; Zaman *et al.,* 2008; Zhao and Jiang, 2013; Liu, Q. Chen, 2015; Zhang *et al.,* 2017)). Aadil *et al.* (2013) studies a stochastic SIRS epidemic model with general incidence rate in a population of varying size and then characterizes the qualitative dynamics of a stochastic SIRS epidemic model (see (Lahrouz and Settati,, 2014)). Zhao and Jiang (2014) investigate the dynamics of a stochastic SIRS epidemic model with saturated incidence. Liu and Chen (2015) analysis of the deterministic and stochastic SIRS epidemic models with nonlinear incidence. Chang *et al.* (2017) presents a novel stochastic SIRS epidemic model with two different saturated incidence rates. Cai *et al.* (2017) investigate a stochastic SIRS epidemic model with nonlinear incidence rate. Chen and Kang (2016) discuss the asymptotic behavior of a stochastic vaccination model with backward bifurcation. Similarly to (2016), we assume that the environmental influence on the individuals is described by stochastic perturbations and it is proportional to each state , S(t), I(t), R(t) . Then, we formulate the stochastic model by introducing the multiplicative noise terms into the deterministic system (1.1) , which is used to model the interaction between the individuals and the environment. Hence, we obtain the following SIRS epidemic model with nonlinear incidence and transfer from infectious to susceptible:

$$\begin{cases} dS = [\lambda - \mu S - Sf(I) + \gamma_1 I + \delta R]dt + \sigma_1 S dB_1(t), \\ dI = [Sf(I) - (\mu + \gamma_1 + \gamma_2 + \alpha)I]\ dt\ + \sigma_2 I dB_2(t), \\ dR = [\gamma_2 I - (\mu + \delta)R]dt + \sigma_3 R dB_3(t). \end{cases} \tag{1.4}$$

where $B_1(t), B_2(t), B_3(t)$ are independent standard Brownian motions, and $\sigma_1, \sigma_2, \sigma_3$ are the intensities of the standard Gaussian white noises, respectively.

This paper is organized as follows. In Section 2, we prove that there is a unique global positive solution of system (1.4) by the way mentioned in [24, 25]. In section3, we show that the disease goes to extinction exponentially under certain conditions. In section 4, we establish sufficient conditions for the existence of an ergodic stationary distribution. In section 5, some conclusion is given.

Existence and uniqueness of positive solution: In order to investigate the dynamical behavior of a population model, the first concern thing is whether the solution is positive and global existence. Hence, in this section, we first show that the solution of system (1.4) is global and positive.

Theorem 2.1. There is a unique solution $(S(t), I(t), R(t))$ of system (1.4) on $t \geq 0$ for any initial value $(S(0), I(0), R(0))$ and the solution will remain in $\mathbb{R}_+^3$ with probability one, namely, $(S(t), I(t), R(t)) \in R_+^3$ for all almost surely $t \geq 0$ almost surely.

The proof of this theorem is similar to the proof in literature (Zhang *et al.,* 2014) so it is omitted.

Extinction of the disease

In this section, we will investigate the condition for the extinction of disease. For simplicity, define

$$\langle X(t) \rangle = \frac{1}{t}\int_0^t X(s)ds, \quad R_s = R_0 - \frac{\sigma_2^2}{2(\mu + \gamma_1 + \gamma_2 + \alpha)}$$

Here, R_0 is the eproductive number of the deterministic model (1.1).

Lemma 3.1. (see [26]) Let $M = \{M_t\}_{t \geq 0}$ be a real-value continuous local martingale vanishing at $t = 0$. Then

$$\lim_{t\to\infty} \langle M, M \rangle_t = \infty \text{ a.s } \Rightarrow \lim_{t\to\infty} \frac{M_t}{\langle M, M \rangle_t} = 0, \text{ a.s}$$

and also

$$\limsup_{t\to\infty} \frac{\langle M, M \rangle_t}{t} < \infty \text{ a.s } \Rightarrow \lim_{t\to\infty} \frac{M_t}{t} = 0, \text{ a.s.}$$

Lemma 3.2. Let $(S(t), I(t), R(t))$ be the solution of system (1.4) with initial value $(S(0), I(0), R(0)) \in \mathbb{R}_+^3$, then

$$\limsup_{t\to\infty} \left(S(t) + I(t) + R(t) \right) < \infty, \quad \text{a. s.} \qquad (3.1)$$

$$\lim_{t\to\infty} \frac{S(t)}{t} = 0, \ \lim_{t\to\infty} \frac{I(t)}{t} = 0, \ \lim_{t\to\infty} \frac{R(t)}{t} = 0, \ \text{a. s.} \qquad (3.2)$$

And

$$\lim_{t\to\infty} \frac{\int_0^t S(u)dB_1(u)}{t} = 0, \ \lim_{t\to\infty} \frac{\int_0^t I(u)dB_2(u)}{t} = 0, \ \lim_{t\to\infty} \frac{\int_0^t R(u)dB_3(u)}{t} = 0, \ \text{a. s.} \quad (3.3)$$

Proof. By the system (1.4), we have

$$d(S + I + R) = \lambda - \mu(S + I + R) - \alpha I + \sigma_1 SdB_1 + \sigma_2 IdB_2 + \sigma_3 RdB_3 \qquad (3.4)$$

Solving this equation, we get

$$\begin{aligned}
S(t)+I(t)+R(t) &= \frac{\lambda}{\mu} + (S(0)+I(0)+R(0) - \frac{\lambda}{\mu})e^{-\mu t} - \alpha\int_0^t I(u)e^{-\mu(t-u)}dt \\
&\quad + \sigma_1\int_0^t S(u)e^{-\mu(t-u)}dB_1(u) + \sigma_2\int_0^t I(u)e^{-\mu(t-u)}dB_2(u) + \sigma_3\int_0^t R(u)e^{-\mu(t-u)}dB_3(u) \\
&\leq \frac{\lambda}{\mu} + (S(0)+I(0)+R(0) - \frac{\lambda}{\mu})e^{-\mu t} + \sigma_1\int_0^t S(u)e^{-\mu(t-u)}dB_1(u) \\
&\quad + \sigma_2\int_0^t I(u)e^{-\mu(t-u)}dB_2(u) + \sigma_3\int_0^t R(u)e^{-\mu(t-u)}dB_3(u) \qquad (3.5) \\
&= X(0) + A(t) - Q(t) + M(t) \\
&:= X(t)
\end{aligned}$$

where

$$X(0) = S(0) + I(0) + R(0)$$
$$A(t) = \frac{\lambda}{\mu} - e^{-\mu t}$$
$$Q(t) = (S(0) + I(0) + R(0))(1 - e^{-\mu t})$$
$$M(t) = \sigma_1 \int_0^t S(u) e^{-\mu(t-u)} dB_1(u) + \sigma_2 \int_0^t I(u) e^{-\mu(t-u)} dB_2(u) + \sigma_3 \int_0^t R(u) e^{-\mu(t-u)} dB_3(u)$$

Clearly, $M(t)$ is a continuous local martingale with $M(0)=0$. It is clear that $A(t)$ and $Q(t)$ are continuous adapted increasing processes on $t \geq 0$ with $A(0)=Q(0)$. By Theorem 3.9 in (Mao, 1997), we obtain that

$$\limsup_{t\to\infty} \left(S(t) + I(t) + R(t)\right) < \infty, \quad \text{a. s.} \qquad (3.6)$$

Thus, the conclusion (3.1) is true. Obviously, according to (3.6), we have that

$$\lim_{t\to\infty} \frac{S(t)}{t} = 0, \ \lim_{t\to\infty} \frac{I(t)}{t} = 0, \ \lim_{t\to\infty} \frac{R(t)}{t} = 0, \quad \text{a. s.}$$

Set

$$M_1(t) = \int_0^t S(u) dB_1(u), \ \ M_2(t) = \int_0^t I(u) dB_2(u), \ \ M_3(t) = \int_0^t R(u) dB_3(u)$$

Since the quadratic variations, we have

$$\langle M_1(t), M_1(t)\rangle = \int_0^t S^2(u) du \leq \left(\sup_{t\geq 0} S^2(t)\right) t$$

By the large number theorem for martingale (see Lemma 3.1) and (3.6), we get

$$\lim_{t\to\infty} \frac{\int_0^t S(u) dB_1(u)}{t} = 0, \quad \text{a. s.}$$

Similarly, we can also get

$$\lim_{t\to\infty} \frac{\int_0^t I(u) dB_2(u)}{t} = 0, \quad \lim_{t\to\infty} \frac{\int_0^t R(u) dB_3(u)}{t} = 0. \qquad \text{a.s.}$$

Thus, the conclusion (3.3) is proved. Hence this finishes the proof Lemma 3.2

Theorem 3.1. Let $(S(t), I(t), R(t))$ be the solution of system (1.4) on $t \geq 0$ for any initial value $(S(0), I(0), R(0)) \in \mathbb{R}_+^3$. If $R_S < 1$, then

$$\limsup_{t\to\infty} \frac{\log I(t)}{t} < (\mu + \gamma_1 + \gamma_2 + \alpha)(R_s - 1) < 0, \qquad \text{a.s}$$

Moreover,

$$\lim_{t\to\infty} \langle S(t)\rangle = \frac{\lambda}{\mu}, \qquad \lim_{t\to\infty} \langle R(t)\rangle = 0, \quad \text{a.s}$$

Proof. An integrating both sides from 0 to t of system (1.4) yields

$$\frac{S(t) - S(0)}{t} + \frac{I(t) - I(0)}{t} + \frac{\delta}{\mu + \delta} \frac{R(t) - R(0)}{t} = \lambda - \mu \langle S(t)\rangle - (\mu + \alpha + \frac{\gamma_2 \mu}{\mu + \delta}) \langle I(t)\rangle + \frac{M(t)}{t}$$

Where

$$M(t)=\frac{\sigma_1}{t}\int_0^t SdB_1(t)+\frac{\sigma_2}{t}\int_0^t IdB_2(t)+\frac{\sigma_3\delta}{(\mu+\delta)t}\int_0^t RdB_3(t)$$

note (3.3), we have $\lim_{t\to\infty}\frac{M(t)}{t}=0$. We compute that

$$\langle S(t)\rangle=\frac{\lambda}{\mu}-\frac{(\mu+\alpha)(\mu+\delta)+\gamma_2\mu}{\mu(\mu+\delta)}\langle I(t)\rangle+\frac{M(t)}{\mu t}+\varphi(t) \tag{3.7}$$

Where $\varphi(t)=\frac{1}{\mu}(-\frac{S(t)-S(0)}{t}-\frac{I(t)-I(0)}{t}-\frac{\delta}{\mu+\delta}\frac{R(t)-R(0)}{t})$. according to Lemma 3.2,

$$\lim_{t\to\infty}\varphi(t)=0 \tag{3.8}$$

Applying $Ito's$ formula to system (1.4), we have

$$\begin{aligned} d\log I(t)&=\left(\frac{\beta S}{1+I}-(\mu+\gamma_1+\gamma_2+\alpha+\frac{\sigma_2^2}{2})\right)dt+\sigma_2 dB_2(t) \\ &\le[\beta S-(\mu+\gamma_1+\gamma_2+\alpha+\frac{\sigma_2^2}{2})]dt+\sigma_2 dB_2(t)\end{aligned} \tag{3.9}$$

Integrating this from 0 to t and dividing t on the both sides and substituting (3.7) into the last inequality of (3.9), we have

$$\begin{aligned} \frac{\log I(t)}{t}&\le\beta\langle S(t)\rangle-(\mu+\gamma_1+\gamma_2+\alpha+\frac{\sigma_2^2}{2})+\frac{\log I(0)}{t}+\frac{\sigma_2 B_2(t)}{t} \\ &=\frac{\beta\lambda}{\mu}-(\mu+\gamma_1+\gamma_2+\alpha+\frac{\sigma_2^2}{2})-\frac{\beta\{(\mu+\alpha)(\mu+\delta)+\gamma_2\mu\}}{\mu(\mu+\delta)}\langle I(t)\rangle \\ &\quad+\frac{\log I(0)}{t}+\frac{\beta M(t)}{\mu t}+\beta\varphi(t)+\frac{\sigma_2 B_2(t)}{t} \\ &\le\frac{\beta\lambda}{\mu}-(\mu+\gamma_1+\gamma_2+\alpha+\frac{\sigma_2^2}{2})+\beta\varphi(t)+\frac{\beta M(t)}{\mu t}+\frac{\log I(0)}{t}+\frac{\sigma_2 B_2(t)}{t}\end{aligned} \tag{3.10}$$

where $\frac{\sigma_2 B_2(t)}{t}$ is a local continuous martingale, according to the large number theorem for martingales (see Lemma 3.1), we have

$$\lim_{t\to\infty}\frac{\sigma_2 B_2(t)}{t}=0, \text{ a.s.}$$

It follows from (3.10), taking the limit superior of both sides, if $R_S<1$, we obtain

$$\limsup_{t\to\infty}\frac{\log I(t)}{t}\le\frac{\beta\lambda}{\mu}-(\mu+\gamma_1+\gamma_2+\alpha+\frac{\sigma_2^2}{2})=(\mu+\gamma_1+\gamma_2+\alpha)(R_s-1)<0$$

which implies

$$\lim_{t\to\infty}I(t)=0, \quad \text{a.s} \tag{3.11}$$

According to (3.7), (3.8) and (3.11), we obtain

$$\lim_{t\to\infty}\langle S(t)\rangle=\frac{\lambda}{\mu}, \text{ a. s.}$$

According to the last equation of the system (1.4) is an asymptotically differential system with limit system

$$\frac{R(t)-R(0)}{t}=\gamma_2\langle I(t)\rangle-(\mu+\delta)\langle R(t)\rangle+\frac{\int_0^t\sigma_3 RdB_3(t)}{t}$$

According to Lemma 3.2, we obtain

$$\lim_{t\to\infty}\langle R(t)\rangle=0,\ \text{a. s.}$$

This finishes the proof of Theorem 3.1.

Remark 3.1. From Theorem 3.1, we can know that the disease will die out exponentially.

Stationary distribution

In this section, we study the existence of a unique stationary distribution of the system (1.4).

Lemma 4.1. [28] The Markov process $X(t)$ has a unique stationary distribution $m(\cdot)$ if there exists a bounded domain $U\in\mathbb{R}^d$ with regular boundary such that its closure $\overline{U}\in\mathbb{R}^d$ having the following properties:

(i) There exist some $i=1,2,\cdots n$ and positive constant η such that $a_{ii}\geq\eta$ for any $x\in U$.
(ii)There is a nonnegative C^2-function $\phi(x)$ and a neighborhood U such that for some constants $K>0, L\phi(x)<-K, x\in\mathbb{R}^d\setminus U$. Moreover, if $f(\cdot)$ is a function integrable with respect to the measure π, then $P(\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X^x(t))dt=\int_{\mathbb{R}^d} f(X^x(t))m(dx))=1,$ for all $x\in\mathbb{R}^d$.

Theorem 4.1. If $R_0>1$, $\alpha^2<4(\mu+\alpha-\sigma_2^2)(\mu-\sigma_3^2)$, and

$$\begin{aligned}&0<\sigma_1^2<\mu,\\&0<\sigma_3^2<\mu-\rho\alpha,\\&0<\sigma_2^2<\frac{1}{c_2+1}[(\mu+\alpha)(c_2+1)+c_2\alpha-\frac{\alpha}{4\rho}],\end{aligned}\tag{4.1}$$

such that

$$\eta<\min\{(\mu-\sigma_1^2)(c_1+1)S^{*2},\ [(\mu+\alpha-\sigma_2^2)(c_2+1)+c_2\alpha-\frac{\alpha}{4\rho}]S^{*2},\ (\mu-\sigma_3^2-\rho\alpha)R^{*2}\}\tag{4.2}$$

then, for any initial value $(S(0),I(0),R(0))\in\mathbb{R}_+^3$ there is a stationary distribution $\pi(\cdot)$ for system (1.4) and it has an ergodic property, where $\eta=\frac{c_1I^*\sigma_2^2}{2}+(\sigma_1^2S^{*2}+\sigma_2^2I^{*2})(c_2+1)+\sigma_3^2R^{*2}$, c_1, c_2, ρ is positive constant to be specified later, $E^*=(S^*,\ I^*,\ R^*)$ is the endemic proportion equilibrium of system (1.1).

Proof. If $R_0>1$, there is an endemic proportion equilibrium $E^*=(S^*,\ I^*,\ R^*)$ of system (1.1), so

$$\begin{aligned}&\lambda=\mu S^*+\frac{\beta S^*I^*}{1+I^*}-\gamma_1I^*-\delta R^*,\\&\frac{\beta S^*I^*}{1+I^*}=(\mu+\gamma_1+\gamma_2+\alpha)I^*,\\&\gamma_2I^*=(\mu+\delta)R^*,\end{aligned}\tag{4.3}$$

Define a C^2 − function $W:\mathbb{R}^3_+ \to \mathbb{R}_+$, by

$$\begin{aligned} W(S,I,R) &= c_1(I-I^*-I^*\ln\frac{I}{I^*})+\frac{c_2}{2}[(S-S^*)+(I-I^*)]^2 \\ &+\frac{1}{2}[(S-S^*)+(I-I^*)+(R-R^*)]^2 \\ &:= \mathrm{c}_1W_1+\mathrm{c}_2W_2+W_3 \end{aligned} \tag{4.4}$$

Then we have

$$LW_1 \le \beta(S-S^*)(I-I^*)+\frac{\sigma_2^2 I^*}{2} \tag{4.5}$$

and

$$\begin{aligned} LW_2 &= [(S-S^*)+(I-I^*)][\lambda-\mu S-(\mu+\gamma_2+\alpha)I+\delta R]+\frac{\sigma_1^2S^2}{2}+\frac{\sigma_2^2I^2}{2} \\ &\le -(\mu-\sigma_1^2)(S-S^*)^2-(\mu+\gamma_2+\alpha-\sigma_2^2)(I-I^*)^2-(2\mu+\gamma_2+\alpha)(S-S^*) \\ &\quad \times(I-I^*)+\delta(S-S^*)(R-R^*)+\delta(I-I^*)(R-R^*)+\sigma_1^2S^{*2}+\sigma_2^2I^{*2} \end{aligned} \tag{4.6}$$

$$\begin{aligned} LW_3 &= [(S-S^*)+(I-I^*)+(R-R^*)][\lambda-\mu S-(\mu+\alpha)I-\mu R]+\frac{\sigma_1^2S^2}{2}+\frac{\sigma_2^2I^2}{2}+\frac{\sigma_3^2R^2}{2} \\ &\le -(\mu-\sigma_1^2)(S-S^*)^2-(\mu+\alpha-\sigma_2^2)(I-I^*)^2-(\mu-\sigma_3^2)(R-R^*)^2 \\ &\quad -(2\mu+\alpha)(S-S^*)(I-I^*)-2\mu(S-S^*)(R-R^*)-(2\mu+\alpha)(I-I^*)(R-R^*) \\ &\quad +\sigma_1^2S^{*2}+\sigma_2^2I^{*2}+\sigma_3^2R^{*2} \end{aligned} \tag{4.7}$$

Then

$$\begin{aligned} LW &\le -(\mu-\sigma_1^2)(c_1+1)(S-S^*)^2-[(\mu+\alpha-\sigma_2^2)(c_2+1)+c_2\alpha](I-I^*)^2-(\mu-\sigma_3^2)(R-R^*)^2 \\ &\quad +[c_1\beta-c_2(2\mu+\gamma_2+\alpha)-(2\mu+\alpha)](S-S^*)(I-I^*)+(c_2\delta-2\mu)(S-S^*)(R-R^*) \\ &\quad +[c_2\delta-(2\mu+\alpha)](I-I^*)(R-R^*)+\frac{c_1I^*\sigma_2^2}{2}+(\sigma_1^2S^{*2}+\sigma_2^2I^{*2})(c_2+1)+\sigma_3^2R^{*2} \end{aligned}$$

Choose $c_1=[(2\mu+\alpha)(2\mu+\delta)+2\mu\gamma_2]/\delta\beta$, $c_2=2\mu/\delta$ such that

$$c_1\beta-c_2(2\mu+\gamma_2+\alpha)-(2\mu+\alpha)=0,\quad c_2\delta-2\mu=0$$

Thus

$$\begin{aligned} LW &\le -(\mu-\sigma_1^2)(c_1+1)(S-S^*)^2-[(\mu+\alpha-\sigma_2^2)(c_2+1)+c_2\alpha-\frac{\alpha}{4\rho}](I-I^*)^2 \\ &\quad -(\mu-\sigma_3^2-\rho\alpha)(R-R^*)^2+\eta \end{aligned}$$

where Youngs inequality is used and

$$\eta=\frac{c_1I^*\sigma_2^2}{2}+(\sigma_1^2S^{*2}+\sigma_2^2I^{*2})(c_2+1)+\sigma_3^2R^{*2} \tag{4.10}$$

Since $\dfrac{\alpha}{4(\mu+\alpha-\sigma_2^2)(\frac{2\mu}{\delta}+1)}<\rho<\dfrac{\mu-\sigma_3^2}{\alpha}$, we can let ρ satisfied

$$\alpha^2 < 4(\mu+\alpha-\sigma_2^2)(\frac{2\mu}{\delta}+1)(\mu-\sigma_3^2),$$

This implies

$$[(\mu+\alpha-\sigma_2^2)(\frac{2\mu}{\delta}+1)+\frac{2\mu}{\delta}\alpha-\frac{\alpha}{4\rho}]>0,\quad \mu-\sigma_3^2-\rho\alpha>0.$$

Note that

$$\eta<\min\{(\mu-\sigma_1^2)(c_1+1)S^{*2},\ [(\mu+\alpha-\sigma_2^2)(c_2+1)+c_2\alpha-\frac{\alpha}{4\rho}]I^{*2},\ (\mu-\sigma_3^2-\rho\alpha)R^{*2}\}.$$

Thus the ellipsoid

$$-(\mu-\sigma_1^2)(c_1+1)(S-S^*)^2-[(\mu+\alpha-\sigma_2^2)(c_2+1)+c_2\alpha-\frac{\alpha}{4\rho}](I-I^*)^2-(\mu-\sigma_3^2-\rho\alpha)(R-R^*)^2+\eta=0$$ lies

entirely in $\mathbb{R}^3_+$. We can take U as any neighborhood of the ellipsoid such that $\overline{U}\in\mathbb{R}^3_+$, $\overline{U}$ where is the closure of U. Thus, we have $LW(S,I,R)<0$ for $(S,I,R)\in\mathbb{R}^3_+\setminus U$, which implies that condition (ii) in Lemma 4.1 is satisfied. On the other hand, for system (1.4), the diffusion matrix is

$$A=\begin{pmatrix}\sigma_1^2S^2 & 0 & 0\\ 0 & \sigma_2^2I^2 & 0\\ 0 & 0 & \sigma_3^2R^2\end{pmatrix}$$

Choose $M=\min_{(S,\ I,\ R)\in\bar{U}\subset R_+^3}\{\sigma_1^2S^2,\ \sigma_2^2I^2,\ \sigma_3^2R^2\}$, we can obtain that,

$$\sum_{i,j=1}^{3}a_{ij}(S,\ I,\ R)\xi_i\xi_j=\sigma_1^2S^2\xi_1^2+\sigma_2^2I^2\xi_2^2+\sigma_3^2R^2\xi_3^2\geq M|\xi|^2,\ (S,\ I,\ R)\in\bar{U},\ \xi=(\xi_1,\ \xi_2,\ \xi_3)\in R_+^3,$$

Then the condition (i) in Lemma 4.1 is satisfied. Thus the system (1.4) has a stationary distribution $\pi(\cdot)$ and is ergodic.

Conclusion

This paper studied that asymptotic behaviors of an stochastic SIRS model with nonlinear incidence rate and transfer from infectious to susceptible. First of all, we analyze that system (1.4) has a unique global positive solution with the initial value. Then, we found that stochastic perturbation in the environment can lead the disease to extinction if $R_s<1$,

In addition, we establish sufficient conditions for the system (1.4) has a stationary distribution $\pi(\cdot)$ and is ergodic as the fluctuation is sufficiently small. Comparing R_0 with R_s, it shows that stochastic disturbances in the environment has an inhibitory effect on the transmission of infectious diseases. Some interesting topics deserve further investigations. On the other hand, we just got sufficient conditions the disease extinction, we will further study the necessary condition of disease extinction. We leave these works for the future.

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