Available Online at http://www.journalajst.com

ISSN: 0976-3376

ASIAN JOURNAL OF SCIENCE AND TECHNOLOGY

Asian Journal of Science and Technology
Vol. 09, Issue, 10, pp.8875-8890, October, 2018

REVIEW ARTICLE

POLYNOMIAL APPROXIMATION ON UNBOUNDED SUBSETS AND SOME APPLICATIONS

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ARTICLE INFO

Article History:
Received 25th July, 2018
Received in revised form
28th August, 2018
Accepted 14th September, 2018
Published online 30th October, 2018

Key words:
Polynomial approximation,
Markov moment problem,
Invariance of the unit ball,
Bounded linear operators.

ABSTRACT

This review paper deals with recalling some applications of polynomial approximation results to Markov moment problem as well as to invariance of the unit ball of L^1 -type spaces, with respect to some bounded linear operators. Polynomial approximation on special unbounded subsets is mainly discussed. One solves partially the difficulty created by the fact that in several real dimensions positive polynomials are not sums of squares. Most of our characterizations are expressed in terms of signatures of products of quadratic mappings. Main such results were published by the authors in the period 2007-2015. Some applications of these theorems, which have been published more recently, are also recalled. Earlier results on extension of linear operators with one or two constraints are applied as well.
AMS: 41A10, 47A57, 47A63

Citation: ***Olteanu, O. and Mihăilă, J.M.*, 2018.** "Polynomial approximation on unbounded subsets and some applications", *Asian Journal of Science and Technology*, 09, (10), 8875-8890.

1. INTRODUCTION

The moment problem is an interpolation problem with one or two constraints. The "lower" constraint is usually the positivity of the linear solution. Sometimes, this makes possible representing the solution by means of a positive scalar or vector measure. The "upper" constraint could refer to the continuity of the solution and "measures" its norm. The latter constraint appears in Markov moment problem (Krein and Nudelman, 1977; Lemnete-Ninulescu and Olteanu, 2017; Mihăilă*et al.,* 2007; Mihăilă*et al.,* 2007; Olteanu and Olteanu, 2007, 2009; Olteanu, 1991; Olteanu, 1996; Olteanu, 2013; Olteanu, 2014; Olteanu, 2015). In the classical moment problem (Akhiezer, 1965), the values of the solution at polynomials are given, and the positivity of the solution is required. The first question is to find and prove necessary and sufficient conditions for the existence of a linear extension from the subspace of polynomials to a larger function space, such that some constraints to be verified. The next two problems are establishing eventually the uniqueness and constructing the solution. So, the moment problem is an extension problem of a linear functional (or operator), with one or two constraints. In the case when our function space is the space of all real continuous functions on a compact subset $K \subset \mathbb{R}^n$, one applies the classical Weierstrass approximation theorem. Thus, in this case, the linear continuous solution (if any) is unique, having (given) prescribed values (called moments) at basic polynomials $t^k = t_1^{k_1} \cdots t_n^{k_n}, k_j \in \mathbb{N}, j \in \{1, \dots, n\}, t = (t_1, \dots, t_n) \in K, k = (k_1, \dots, k_n) \in \mathbb{N}^n$. If one replaces the compact K by an unbounded closed subset, more powerful polynomial approximation results are required in order to deduce the density of the subspace of polynomials in some function spaces, such as L^1- type spaces. This is the motivation of recalling the results from section 2. The moment problem, as well as other questions in analysis is related to the analytic form of positive polynomials on some closed subsets, in terms of sums of squares of some other polynomials (Akhiezer, 1965; Berg *et al.,* 1979; Cassier, 1984; Choquet, 1962; Lemnete-Ninulescu and Olteanu, 2017; Mihăilă *et al.,* 2007; Olteanu and Olteanu, 2007, 2009; Putinar, 1993; Schmüdgen, 1991; Schmüdgen, 2017; Vasilescu, 2003). Such representations allow writing the values of a linear functional at a positive polynomial in terms of quadratic mappings or products of quadratic mappimgs, the latter generalizing classical results from the one dimensional case to several dimensions. Such techniques are applied in both sections 3 and 4. In this article, quadratic forms appearing in the classical moment problem are replaced by quadratic vector-valued mappings.

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The rest of this paper is organized as follows. Section 2 is devoted to polynomial approximation on unbounded subsets. In Section 3, a few aspects of the Markov moment problem are pointed out. In Section 4, invariance of the unit ball in L^1- type spaces with respect to some bounded linear operators is discussed.

2. POLYNOMIAL APPROXIMATION ON UNBOUNDED SUBSETS

The main results of this section were stated and proved in (Mihăilă *et al.,* 2007; Olteanu and Olteanu, 2007; Olteanu, 2014; Olteanu, 2015) and recalled in two other review papers. In these latter works, the poofs are not presented any more. Here we give the proofs for the multidimensional case, which is the most interesting for the purpose of the present work.These results will be applied in the sequel (see subsection 3.2 below).

Lemma 2.1. *Let* $\psi:[0,\infty)\to R_+$ *be a continuous function, such that* $\lim_{t\to\infty}\psi(t)\in R_+$ *exists. Then there is a decreasing sequence* $(h_l)_l$ *in the linear hull of the functions*

$$\varphi_k(t)=\exp(-kt),\quad k\in \mathrm{N},\quad t\geq 0,$$

Such that $h_l(t)>\psi(t),\ t\geq 0,\ l\in \mathrm{N},\ \lim h_l=\psi$ *uniformly on* $[0,\infty)$. *There exists a sequence of polynomial functions* $(\tilde{p}_l)_{l\in\mathbb{N}},\tilde{p}_l\geq h_l>\psi, \lim\tilde{p}_l=\psi$, *uniformly on compact subsets of* $[0,\infty)$.

The idea of the proof is to add the ∞ point and to apply the Stone-Weierstrass Theorem to the subalgebra generated by the functions $\exp(-mt)$, $m\in Z_+$. Then one uses for each such exp – function suitable majorizing or minorizing polynomial - sums, as well as the obvious equality

$$\exp(s)-\left(1+\frac{s}{1!}+\frac{s^2}{2!}+\cdots+\frac{s^m}{m!}\right)=\frac{\exp(s)}{m!}\int_0^s \exp(-t)\cdot t^m\,dt,\quad m\in Z_+.$$

Lemma 2.2. *Let* ν *be a M-determinate positive regular measure on* $[0,\infty)$, *with finite moments of all natural orders. If* ψ, $(\tilde{p}_l)_l$ *are as in Lemma 2.1, then there exists a subsequence* $(\tilde{p}_{l_m})_m$, *such that* $\tilde{p}_{l_m}\to\psi$ *in* $L^1_\nu([0,\infty))$ *and uniformly on compact subsets. In particular, it follows that the positive cone* P_+ *of positive polynomials is dense in the positive cone* $(L^1_\nu([0,\infty)))_+$ *of* $L^1_\nu([0,\infty))$

Lemma 2.3. *For any simple function of the form*

$$s=\sum_{j=1}^{N}\alpha_j\chi_{I_j},\quad \alpha_j\geq 0,\quad j=1,\ldots,N,\quad I_j\subset R_+,\quad j=1,\ldots,N,$$

I_j *being intervals, there exists a sequence of polynomials* $(p_m)_m$, $p_m\to s$ *in* $L^1_\nu([0,\infty))$, $p_m>s$, $\forall m$, *where* ν *is a M-determinate positive regular Borel measure with finite moments of all natural orders. If s is extended to an even function over R, then the polynomials* p_m, $m\in N$ *are restrictions to the positive semi axis of even polynomials on R.*

We recall that a $M-$determinate measure is, by definition, uniquely determinate by its moments, or, equivalently, by its values on polynomials.

The novelty of Lemma 2.2 consists in the approximation from above, which implies the positivity of the approximating polynomials. For positive regular Borel measures, Luzin's theorem and approximation by continuous compactly supported functions do work. If we additionally assume that such a measure is M-determinate, then polynomial approximation proved in lemma 2.4 from below holds too. One can give a proof of Lemma 2.2, which is different from that of lemma 2.4 (see (Mihăilă *et al.,* 2007; Olteanu and Olteanu, 2007, 2009)).

Lemma 2.4 is important in itself, due to its generality. The proof is based on a Hahn-Banach argument, also using elements of measure theory. Now let us recall some notations. If *A* is a locally compact topological space (for example an unbounded closed subset of R^n) then $C_0(A)$ is the space of all continuous real functions on *A*, vanishing at infinity. By $C_c(A)$ one denotes the space of all real continuous compactly supported functions on *A*, having their supports contained in *A*. If *Y* is an ordered vector space, then Y_+ denotes the positive cone of *Y*. The next polynomial approximation result on an unbounded (closed) subset will be applied in the sequel.

Lemma 2.4. *Let $A \subset R^n$ be an unbounded closed subset, and ν an M-determinate positive regular Borel measure on A, with finite moments of all natural orders. Then for any $x \in (C_0(A))_+$, there exists a sequence $(p_m)_m$, $p_m \in P_+$, $p_m \geq x$, $p_m \to x$ in $L^1_\nu(A)$. In particular, we have:*

$$\lim \int_A p_m(t) d\nu = \int_A x(t) d\nu,$$

P_+ is dense in $\left(L^1_\nu(A)\right)_+$, and P is dense in $L^1_\nu(A)$.

Proof. Let consider the sublattice $X \subset L^1_\nu(A)$ of all function ψ such that $|\psi|$ is dominated by some polynomial p on A. To prove the assertions of the statement, it is sufficient to show that for any $x \in (C_0(A))_+$, we have

$$Q_1(x) := \inf\left\{\int_A p(t) d\nu;\ \ p \geq x,\ \ p \in P\right\} = \int_A x(t) d\nu.$$

Obviously, one has

$$Q_1(x) \geq \int_A x(t)\, d\nu \tag{2.1}$$

To prove the converse, we define the linear form

$$F_0 : X_0 = P \oplus Sp\{x\} \to R,\ \ F_0(p + ax) := \int_A p(t) d\nu + a \cdot Q_1(x),\ \ \ p \in P,\ \ \ a \in R.$$

Next we show that F_0 is positive on X_0. In fact, for $a < 0$, one has (from the definition of Q_1, which is a sublinear functional on X)

$$p + ax \geq 0 \Rightarrow p \geq -ax \Rightarrow (-a)Q_1(x) = Q_1(-ax) \leq \int_A p(t) d\nu \Rightarrow F_0(p + ax) \geq 0.$$

If $a \geq 0$, we infer that

$$0 = Q_1(0) = Q_1(ax + (-ax)) \leq aQ_1(x) + Q_1(-ax) \Rightarrow$$

$$\int_A p(t) d\nu \geq Q_1(-ax) \geq -aQ_1(x) \Rightarrow F_0(p + ax) \geq 0.$$

Whence, in both possible cases, $x_0 \in (X_0)_+ \Rightarrow F_0(x_0) \geq 0$. Since X_0 contains the space of polynomials functions, which is a majorizing subspace of X, there exists a linear positive extension $F : X \to R$ of F_0 (cf. Theorem 3.1.5 below), that is continuous on $C_0(A)$, with respect to the sup-norm. Therefore, F has a representation by means of a positive Borel regular measure μ on A, such that

$$F(x) = \int_A x(t) d\mu,\ \ \ x \in C_0(A).$$

Let $p \in P_+$ be a nonnegative polynomial function. There is a nondecreasing sequence $(x_m)_m$ of continuous nonnegative function with compact support, such that $x_m \uparrow p$, point wise on A. Positivity of F and Lebesgue dominated convergence theorem for μ yield

$$\int_A p d\nu = F(p) \geq \sup F(x_m) = \sup \int_A x_m(t) d\mu = \int_A p d\mu,\ \ \ p \in P_+.$$

Thanks to Haviland theorem, there exists a positive Borel regular measure λ on A, such that

$$\lambda(p) = \nu(p) - \mu(p) \Leftrightarrow \nu(p) = \lambda(p) + \mu(p),\ \ \ p \in P.$$

Since ν is assumed to be M-determinate, it follows that

$$\nu(B)=\mu(B)+\lambda(B),$$

for any Borel subset B of A. From this last assertion, approximating each $x\in(L^1_\nu(A))_+$ by a nondecreasing sequence of nonnegative simple functions, and also using Lebesgue convergence theorem, one obtains firstly for positive functions, then for arbitrary ν -integrable functions x:

$$\int_A x d\nu=\int_A x d\mu+\int_A x d\lambda,\quad x\in L^1_\nu(A).$$

In particular, we must have

$$\int_A x d\nu \geq F(x)=F_0(x)=Q_1(x) \tag{2.2}$$

Now (2.1) and (2.2) conclude the proof.□

Note the polynomials appearing in the preceding lemma 2.4 are nonnegative on A. However, this does not solve our problem, because the form of positive (or nonnegative) polynomials on an arbitrary unbounded closed subset A is not known. Therefore, the connection to the polynomials of one variable would be convenient. The next result makes the connection of nonnegative compactly supported functions of several variables with the sums of products of squares of some other polynomials, in one-dimensional variable. Precisely, the following result holds.

Lemma 2.5. *Let $\nu=\nu_1\times\nu_2\times\cdots\times\nu_n$ be a product of nM − determinate positive regular Borel measures on R, with finite moments of all natural orders. Then we can approximate any nonnegative continuous compactly supported function in $X:=L^1_\nu(R^n)$ by means of sums of tensor products $p_1\otimes p_2\otimes\cdots\otimes p_n$, p_j positive polynomial on the real line, in variable t_j, $j=1,\ldots,n$.*

Proof. If K is the support of a continuous compactly supported nonnegative function $f\in C_c(R^n)$, then

$$K\subset K_1\times K_2\times\cdots\times K_n,\quad K_j=pr_j(K),\quad j=1,\ldots,n.$$

Consider a parallelepiped

$$S_n=[a_1,b_1]\times\cdots\times[a_n,b_n],\quad a_j=\inf K_j,\quad b_j=\sup K_j,\quad j=1,\ldots,n$$

containing the above Cartesian product of compacts and apply approximation of f on S_n by the corresponding Bernstein polynomials in nv ariables. Namely, the explicit form of the Bernstein polynomials is

$$B_m(f)(t_1,\ldots,t_n)=$$
$$=\sum_{k_1\in\{0,\ldots,m\}}\cdots\sum_{k_n\in\{0,\ldots,m\}}p_{mk_1}(t_1)\ldots p_{mk_n}(t_n)f\left(a_1+(b_1-a_1)\frac{k_1}{m},\ldots,a_n+(b_n-a_n)\frac{k_n}{m}\right),$$
$$p_{mk_j}(t_j)=\binom{m}{k_j}\left(\frac{t_j-a_j}{b_j-a_j}\right)^{k_j}\left(\frac{b_j-t_j}{b_j-a_j}\right)^{m-k_j},\quad t_j\in[a_j,b_j],\quad j=1,\ldots,n,$$
$$B_m(f)\to f,\ m\to\infty.$$

Each term of such a polynomial is a tensor product $p_1\otimes p_2\otimes\cdots\otimes p_n$, of positive polynomials in each variable, on the projection $pr_j(S_n)$, $j=1,\ldots,n$. Extend each p_j such that it vanishes outside $pr_j(S_n)$, applying then Luzin's theorem, $j=1,\ldots,n$. This procedure does not change the values of p_j on K_j. One obtains approximation by sums of tensor products of positive continuous functions with compact support, in each variable t_j, $j=1,\ldots,n$. The approximating process holds in L^1 norm, and uniformly on K. Now application of lemma 2.4 to $n=1$, $A=R$, leads to approximation of each such function in each separate variable by a dominating (positive) polynomial, in the space $L^1_{\nu_j}(R)$, $j=1,\ldots,n$. The conclusion follows.□

3. ON MARKOV MOMENT PROBLEM

3.1. Introduction and general extension results

As we have already seen in the Introduction (Section 1), Markov moment problem is an extension type problem for linear functionals or operators, with two constraints. The present chapter represents a part of Chapter I from [9] (see also the references therein). It focuses on characterizing or finding sufficient conditions for the existence of a solution of a Markov moment problem. Sometimes the uniqueness of the solution follows too. To this aim, generalizations of Hahn – Banach type results, polynomial approximation on some special closed unbounded subsets, Krein – Milman theorem and related results are applied. The main related extension results can be found in (Olteanu, 1978, 1983, 1991, 1996). We recall the classical formulation of the moment problem, under the terms of T. Stieltjes, given in 1894-1895 (see the basic book of N.I. Akhiezer for details): find the repartition of the positive mass on the nonnegative semi-axis, if the moments of arbitrary orders k ($k = 0,1,2,\dots$) are given. Precisely, in the Stieltjes moment problem, a sequence of real numbers $(s_k)_{k\geq 0}$ is given and one looks for a nondecreasing real function $\sigma(t)$ ($t \geq 0$), which verifies the moment conditions:

$$\int_0^{\infty} t^k d\sigma = s_k \qquad (k = 0,1,2,\dots).$$

This is a one dimensional moment problem, on an unbounded interval. Namely, is an interpolation problem with the constraint on the positivity of the measure dσ. The existence, the uniqueness and the construction of the solution σ are studied. The present chapter concerns firstly the existence problem. If the interval is replaced by a subset of $\mathbb{R}^n$, we have a multidimensional moment problem. The connection with the positive polynomials and extensions of linear positive functional and operators is quite clear. It was studied by many authors appearing in the references of the papers and books listed in References (see Section 1). If the sequence of the real numbers - moments is replaced by a sequence of operators, we have an operator-valued moment problem. Most of the problems appearing in applications require not only the existence of a positive solution, but also an upper constraint on the solution. This is the Markov moment problem. The upper constraint on the solution controls its norm, while the lower constraint is usually the positivity of the solution. Many of these solutions are unique (we have M-determinate Markov moment problems). One of the most useful earlier results is Lemma of the majorizing subspace (see below). The main problem was to find necessary and sufficient conditions for the existence of a solution of the interpolation problem, preserving sandwich conditions. In this general case, the operators involved in the (convex and respectively concave) constraints are defined on arbitrary convex subsets. Here we recall an answer published firstly in 1991, without losing convexity, but strongly generalizing the classical result. This answer is based on previous results published in (Olteanu, 1978, 1983) (see below). Parts of these generalizations of the Hahn-Banach principle are involved in the present work too. Throughout this first part, X will be a real vector space, Y an order-complete vector lattice, $A, B \subset X$ convex subsets, $W: A \to Y$ a concave operator, $T: B \to Y$ a convex operator, $S \subset X$ a vector subspace, $f: S \to Y$ a linear operator.

Theorem 3.1.1. *Assume that:*

$$f|_{S\cap A} \geq W|_{S\cap A}, \quad f|_{S\cap B} \leq T|_{S\cap B}.$$

The following assertions are equivalent:

(a) *there exists a linear extension* $F: X \to Y$ *of the operator f such that* $F|_A \geq W, F|_B \leq T$;
(b) *there exists* $T_1 : A \to Y$ *convex and* $W_1 : B \to Y$ *concave operator such that for all*

$$(\rho, t, \lambda', a_1, a', b_1, b', v) \in [0,1]^2 \times (0,\infty) \times A^2 \times B^2 \times S,$$

one has

$$(1-t)a_1 - tb_1 = v + \lambda'[(1-\rho)a' - \rho b'] \Rightarrow$$
$$(1-t)T_1(a_1) - tW_1(b_1) \geq f(v) + \lambda'[(1-\rho)W(a') - \rho T(b')].$$

Thus in the last relation, we have a convex operator in the left hand side, and a concave operator in the right hand side. The following result related to the theorem of H. Bauer follows.

Theorem 3.1.2. *Let X be a preordered vector space with its positive cone* X_+, *Y an order complete vector lattice,* $T: X \to Y$ *a convex operator,* $S \subset X$ *a vector subspace,* $f: S \to Y$ *a linear positive operator. The following assertions are equivalent:*

(a) *there exists a linear positive extension* $F: X \to Y$ *of f such that* $F(x) \leq T(x)$, $\forall x \in X$;
(b) $f(s) \leq T(x)$ *for all* $(s,x) \in S \times X$ *such that* $s \leq x$.

Now we can deduce the main results on the abstract moment problem.

Theorem 3.1.3. *Let* $X, Y, T: X \to Y$ *be as in Theorem 3.1.2,* $\{x_j\}_{j\in J} \subset X$, $\{y_j\}_{j\in J} \subset Y$ *given families. The following assertions are equivalent:*

(a) There exists a linear positive operator $F: X \to Y$ *such that*

$$F(x_j) = y_j \ \forall j \in J, \ \ F(x) \le T(x) \ \forall x \in X;$$

(b) For any finite subset $J_0 \subset J$ *and any* $\{\lambda_j\}_{j \in J_0} \subset R$, *we have:*

$$\sum_{j \in J_0} \lambda_j x_j \le x \Rightarrow \sum_{j \in J_0} \lambda_j y_j \le T(x)$$

A clearer sandwich-moment problem variant is the following one.

Theorem 3.1.4. *Let* $X, Y, \{x_j\}_{j \in J}, \{y_j\}_{j \in J}$ *be as in Theorem 1.3 and* $F_1, F_2 \in L(X, Y)$ *two linear operators. The following statements are equivalent:*

(a) There exists a linear operator $F \in L(X, Y)$ *such that*

$$F_1(x) \le F(x) \le F_2(x), \quad \forall x \in X_+, \quad F(x_j) = y_j, \quad \forall j \in J;$$

(b) For any finite subset $J_0 \subset J$ *and any* $\{\lambda_j\}_{j \in J_0} \subset R$, *we have:*

$$\left(\sum_{j \in J_0} \lambda_j x_j = \varphi_2 - \varphi_1, \ \varphi_1, \varphi_2 \in X_+ \right) \Rightarrow \sum_{j \in J_0} \lambda_j y_j \le F_2(\varphi_2) - F_1(\varphi_1).$$

The last result of this subsection is an earlier extension result, called Lemma of the majorizing subspace, for positive linear operators on subspaces in ordered vector spaces (X, X_+), for which the positive cone X_+ is generating $(X = X_+ - X_+)$. Recall that in a such an ordered vector space X, a vector subspace S is called a majorizing subspace if for any $x \in X$, there exists $s \in S$ such that $x \le s$.

Theorem 3.1.5. *Let X be an ordered vector space whose positive cone is generating,* $S \subset X$ *a majorizing vector subspace, Y an order complete vector lattice,* $F_0: S \to Y$ *a linear positive operator. Then* F_0 *has a linear positive extension* $F: X \to Y$ *at least.*

Some of the results of this chapter are applications of the theorems stated above. Most of our proofs involve inequalities. The results stated below have been published in previous articles or books, recently recalled in (Lemnete-Ninulescu and Olteanu, 2017). The latter book concerns also the connection of the moment problem with operator theory and the complex moment problem. The reader who is interested in connections of the moment problem with operator theory can study the works (Fuglede, 1983; Lemnete-Ninulescu, 2017; Putinar, 1993; Schmüdgen, 2017; Vasilescu, 2003).

3.2. Polynomial approximation and Markov moment problem

In the sequel, one applies the results of Sections 2 and 3.1 in order to prove the existence and uniqueness theorems for the solutions of some Markov moment problems on Cartesian products of unbounded intervals (the multidimensional case). The one dimensional case follows as a consequence.

Let $\nu = \nu_1 \times \nu_2 \times \cdots \times \nu_n$, where ν_j, $j = 1, \ldots, n$ are positive Borel regular *M*-determinate measures on *R*, with finite moments of all natural orders. Let

$$\varphi_j(t_1, \ldots, t_n) = t_1^{j_1} \cdots t_n^{j_n}, \quad j = (j_1, \ldots, j_n) \in \mathrm{N}^n, \quad (t_1, \ldots, t_n) \in R^n.$$

Let $X = L_\nu^1(R^n)$, *Y* be an order complete Banach lattice, and $(y_j)_{j \in \mathrm{N}^n}$ a multi-indexed sequence in *Y*.

Theorem 3.2.1. Let $F_2: X \to Y$ *be a positive linear bounded operator. The following statements are equivalent:*

(a) *there exists a unique bounded linear operator* $F: X \to Y$, *such that*

$$F(\varphi_j) = y_j, \quad \forall j \in \mathrm{N}^n,$$

F is between zero and F_2 *on the positive cone of X,* $\| F \| \le \| F_2 \|$;

(b) *for any finite subset* $J_0 \subset \mathrm{N}^n$, *and any* $\{\lambda_j;\ j \in J_0\} \subset \mathrm{R}$, *we have*

$$\sum_{j \in J_0} \lambda_j \varphi_j(t) \geq 0\ \forall t \in \mathrm{R}^n \Rightarrow \sum_{j \in J_0} \lambda_j y_j \in Y_+,$$

for any finite subsets $J_k \subset \mathrm{N}$, $k = 1, \ldots, n$ *and any*

$$\{\lambda_{j_k}\}_{j_k \in J_k} \subset R, \quad k = 1, \ldots, n,$$

one has

$$\sum_{i_1, j_1 \in J_1} \left(\cdots \left(\sum_{i_n, j_n \in J_n} \lambda_{i_1} \lambda_{j_1} \cdots \lambda_{i_n} \lambda_{j_n} y_{i_1 + j_1, \ldots, i_n + j_n} \right) \cdots \right) \leq$$

$$\sum_{i_1, j_1 \in J_1} \left(\cdots \left(\sum_{i_n, j_n \in J_n} \lambda_{i_1} \lambda_{j_1} \cdots \lambda_{i_n} \lambda_{j_n} F_2(\varphi_{i_1 + j_1, \ldots, i_n + j_n} \right) \cdots \right).$$

Proof. We define F_0 on the space of polynomials, such that the moment (interpolation) conditions are accomplished. Conditions (b) say that F_0 is positive on the convex cone of all point wise nonnegative polynomials on R^n and is dominated by F_2 on the convex cone generated by the tensor products of positive polynomials in each separate variable appearing in lemma 2.5. Such polynomials are sums of squares of some other polynomials with real coefficients. Hence, the implication (a) $\Rightarrow$ (b) is obvious. For the converse, let ψ be a nonnegative continuous compactly supported function defined on R^n. By the preceding Lemma 2.5, one approximates ψ on a hyper parallelepiped S_n containing

$$pr_1(\text{support } \psi) \times \cdots \times pr_n(\text{support } \psi)$$

by means of the corresponding Bernstein polynomials in *n* variables, on S_n (Lemma 2.5). Then one approximates ψ by sums of tensor products of positive polynomials on *R*:

$$\sum_{j=0}^{k(m)} p_{m,1,j} \otimes \cdots \otimes p_{m,n,j} \to \psi, \quad m \to \infty,$$

in the space $L^1_\nu(R^n)$. On the other hand, the linear positive operator F_0 has a linear positive extension *F* defined on the space of all integrable functions with their absolute value dominated on R^n by a polynomial (cf. Theorem 3.1.5). This space contains the space of continuous compactly supported functions. Whence, for any linear positive functional h on Y, $h \circ F$ can be represented by a regular positive Radon measure. Moreover, using (b) and applying Fatou's lemma, one obtains:

$$0 \leq h(F(\psi)) \leq \liminf_m (h \circ F)\left(\sum_{j=0}^{k(m)} p_{m,1,j} \otimes \cdots \otimes p_{m,n,j} \right) \leq$$
$$\lim_m (h \circ F_2)\left(\sum_{j=0}^{k(m)} p_{m,1,j} \otimes \cdots \otimes p_{m,n,j} \right) = h(F_2(\psi)), \quad \psi \in (C_c(R^n))_+, \quad h \in Y_+^*. \tag{3.2.1}$$

Assume that

$$F_2(\psi) - F(\psi) \notin Y_+ .$$

Using a separation theorem, it should exist a positive linear continuous functional $h \in Y_+^*$ such that

$$h(F_2(\psi) - F(\psi)) < 0,$$

that is $h(F_2(\psi) < h(F(\psi))$. This relation contradicts (3.2.1). The conclusion is that we must have

$$F(\psi)\le F_2(\psi),\quad \psi\in (C_c(R^n))_+ \;.$$

Then for an arbitrary compactly supported continuous function $g\in C_c(R^n)$ one writes

$$|F(g)|\le F_2(g^+)+F_2(g^-)=F_2(|g|)\Rightarrow \|F(g)\|\le \|F_2\|\cdot\|g\|_1 \;.$$

Consequently, the operator F is positive and continuous, of norm dominated by $\|F_2\|$, on a dense subspace of $L^1_\nu(R^n)$. It has a unique linear extension preserving these properties.□

Let $X=L^1_\nu([0,\infty)\times\cdots\times[0,\infty))$, $\nu=\nu_1\times\cdots\times\nu_n$, ν_l, $l=1,\ldots,n$ being positive M −determinate measures on $[0,\infty)$. Repeating the proof of Theorem 3.2.1, and using the form of positive polynomials on R_+ [1] in terms of sums of squares: $(p_l(t_l)=p_{l,1}^2(t_l)+t_l p_{l,2}^2(t_l),\ t_l\ge 0,\ l=1,\ldots,n)$, one obtains a similar statement for this case. Under the same assumptions on Y, and using the same hypothesis and notations, one obtains the following result.

Theorem 3.2.2. *Let $(y_j)_{j\in N^n}$ be a sequence in Y. The following statements are equivalent*

(a) there exists a unique (bounded) linear operator $F\in B(X,Y)$ such that $F(\varphi_j)=y_j$, $j\in N^n$, F is between zero and F_2 on the positive cone of X, $\|F\|\le\|F_2\|$;

(b) for any finite subset $J_0\subset N^n$ and any $\{\lambda_j; j\in J_0\}\subset R$, we have

$$\sum_{j\in J_0}\lambda_j\varphi_j(t)\ge 0\ \forall t\in R_+^n \Rightarrow \sum_{j\in J_0}\lambda_j y_j\in Y_+ \;;$$

for any finite subsets $J_k\subset N$, $k=1,\ldots,n$ and any $\{\lambda_{j_k}\}_{j_k\in J_k}\subset R$, $k=1,\ldots,n$, one has

$$\sum_{i_1,j_1\in J_1}\left(\cdots\left(\sum_{i_n,j_n\in J_n}\lambda_{i_1}\lambda_{j_1}\cdots\lambda_{i_n}\lambda_{j_n}y_{i_1+j_1+l_1,\ldots,i_n+j_n+l_n}\right)\cdots\right)\le$$

$$\sum_{i_1,j_1\in J_1}\left(\cdots\left(\sum_{i_n,j_n\in J_n}\lambda_{i_1}\lambda_{j_1}\cdots\lambda_{i_n}\lambda_{j_n}F_2(\varphi_{i_1+j_1+l_1,\ldots,i_n+j_n+l_n}\right)\cdots\right),\quad (l_1,\ldots,l_n)\in\{0,1\}^n.$$

Corollary 3.2.1. *Let ν be a positive $M-$ determinate regular Borel measure on $\mathbb{R}$ (with finite moments of all orders), $\varphi_j(t)=t^j, t\in\mathbb{R}, j\in\mathbb{N}, X:=L^1_\nu(\mathbb{R}), Y$ an order complete Banach lattice, $(y_j)_{j\in\mathbb{N}}$ a sequence of given elements in Y, F_2 a given bounded positive linear operator applying X into Y. The following statements are equivalent:*

(a) *there exists a unique bounded linear operator from X into Y such that the interpolation conditions*

$$F(\varphi_j)=y_j, j\in\mathbb{N}$$

are accomplished, F is between zero and F_2 on the positive cone of X, $\|F\|\le\|F_2\|$;

(b) *for any finite subset $J_0\subset\mathbb{N}$ and any $\{\lambda_j\}_{j\in J_0}\subset\mathbb{R}$, one has*

$$0\le\sum_{i,j\in J_0}\lambda_i\lambda_j\, y_{i+j}\le\sum_{i,j\in J_0}\lambda_i\lambda_j\, F_2(\varphi_{i+j})$$

Proof. The implication (a) ⇒ (b)is obvious, due to the properties of F, also observing that

$$\sum_{i,j\in J_0}\lambda_i\lambda_j\, y_{i+j}=F\left(\sum_{i,j\in J_0}\lambda_i\lambda_j\,\varphi_{i+j}\right)=F\left(\left(\sum_{j\in J_0}\lambda_j\varphi_j\right)^2\right)\in Y_+.$$

$$\sum_{i,j\in J_0}\lambda_i\lambda_j y_{i+j} = F\left(\left(\sum_{j\in J_0}\lambda_j\varphi_j\right)^2\right) \le F_2\left(\left(\sum_{j\in J_0}\lambda_j\varphi_j\right)^2\right) = \sum_{i,j\in J_0}\lambda_i\lambda_j F_2(\varphi_{i+j})$$

The converse implication is a consequence of Theorem 3.2.1, also using the fact that any positive polynomial (with real coefficients) on the whole real line is a sum of (two) squares of polynomials. Thus, the first implication (b) from Theorem 3.2.1 is equivalent to the first inequality (b) of the present corollary. This concludes the proof.□

Using a similar remark to that of Corollary 3.2.1, from Theorem 3.2.2 one deduces the following result.

Corollary 3.2.2. *Under the same hypotheses and using the same notations, where one replaces* $\mathbb{R}$ *by* $\mathbb{R}_+$, $X := L^1_\nu(\mathbb{R}_+)$, *the following statements are equivalent:*

(a) *There exists a unique bounded linear operator from* X *into* Y *such that the interpolation conditions*

$$F(\varphi_j) = y_j, j \in \mathbb{N}$$

are accomplished, F *is between zero and* F_2 *on the positive cone of* X, $\|F\| \le \|F_2\|$;

(b) *For any finite subset* $J_0 \subset \mathbb{N}$ *and any* $\{\lambda_j\}_{j\in J_0} \subset \mathbb{R}$, *one has*

$$0 \le \sum_{i,j\in J_0}\lambda_i\lambda_j y_{i+j+l} \le \sum_{i,j\in J_0}\lambda_i\lambda_j F_2(\varphi_{i+j+l}), l \in \{0,1\}$$

In the end of this section, one recalls a well-known important example which might stand for the space Y in the previous results. Let H be an arbitrary complex Hilbert space, $A \in \mathcal{A}$ a linear (bounded) self-adjoint operator acting on H, (where $\mathcal{A}$ is the real vector space of all self-adjoint operators acting on H). Denote

$$Y_1 := \{U \in \mathcal{A}; UA = AU\}, Y = Y(A) := \{V \in Y_1; UV = VU, \forall U \in Y_1\} \tag{3.2.2}$$

Obviously, Ydefined by (3.2.2) is a commutative real operator algebra. It is also an order complete Banach lattice, endowed with the usual order relation: $U \le V \Leftrightarrow \langle Uh, h\rangle \le \langle Vh, h\rangle, \forall h \in H, U, V \in Y$, and operatorial norm (cf. [5], pp. 303-305).

3.3. Extreme points, Markov moment problem and a related inverse problem

The aim of this Section is to recall some ideas from (Olteanu, 2013; Lemnete-Ninulescu and Olteanu, 2017) (see also the references therein). One solves truncated moment problems and one points out their connection to the full moment problem. Let us denote

$$\psi_j(t) = j\cdot t^{j-1}, \quad t\in[0,b], \quad j \in \mathrm{N}\setminus\{0\}.$$

Theorem 3.3.1. *For a given family of numbers* $(m_j)_{j=1}^n$, *consider the following statements:*

(a) *there exists* $h \in L^\infty([0,b])$ *such that*

$$0 \le h(t) \le 1 \; a.e., \quad m_j = j\int_0^b t^{j-1}h(t)dt, \quad j = 1,2,\ldots,n;$$

(b) *for any family of scalars* $(\lambda_j)_{j=1}^n$, *one has*

$$\sum_{j=1}^n \lambda_j m_j \le \sum_{j=1}^n \lambda_j b^j \;;$$

(c) *there exists a Borel subset B such that*

$$\int_B j\cdot t^{j-1}dt = m_j, \quad j = 1,\ldots,n.$$

Then (b) $\Rightarrow$ *(a)* $\Leftrightarrow$ *(c).*

Proof. Let the point (b) be accomplished and assume that

$$\sum_{j=1}^{n} \lambda_j \cdot j \cdot t^{j-1} = \varphi_2 - \varphi_1, \quad \varphi_j \in (L^1([0,b]))_+ .$$

Then integration on $[0,b]$ yields

$$\sum_{j=1}^{n} \lambda_j b^j \le \int_0^b \varphi_2(t)dt = F_2(\varphi_2) \Rightarrow \sum_{j=1}^{n} \lambda_j m_j \le \int_0^b \varphi_2(t)dt \; .$$

Application of Theorem 3.1.4 to $F_1 \equiv 0$, F_2 defined above, leads to the existence of a linear positive form F on $L^1([0,b])$ such that

$$F(\psi_j) = m_j,\ j = 1,\ldots,n, \quad F(\psi) \le \int_0^b \psi \cdot dt, \quad \psi \in (L^1([0,b]))_+ \Rightarrow$$

$$|F(\varphi)| \le F(\varphi^+) + F(\varphi^-) \le \int_0^b |\varphi(t)| \cdot dt, \quad \varphi \in L^1([0,b]).$$

Thus, $(b) \Rightarrow (a)$ is proved. (if the characteristic functions of Borel subsets stand for ψ, then the conclusion (a) follows by measure theory arguments). The implication (a) $\Rightarrow$ (c) is a consequence of equality (15.14) [7] (see also Exercise 2.57 [7]). The set of values for the control function u, namely $[-1,1]$ is replaced by $[0,1]$, the set of values for h, which stands for the control function u. The extreme points of the positive part of the unit ball of $L^\infty([0,b])$ are the characteristic functions of measurable sets. The converse is obvious.□

Corollary 3.3.1. *Under the equivalent conditions (a), (c) of Theorem 3.3.1, there exist sequences*

$$y_{1,n} < x_{1,n} < y_{2,n} < x_{2,n} < \ldots < y_{l,n} < x_{l,n} < \ldots, \quad n \in \mathrm{N},$$

such that the following relations hold

$$m_k = \inf_{n \in \mathrm{N}} \left(\sum_{j=1}^{\infty} \left(x_{j,n}^k - y_{j,n}^k \right) \right), \quad k = 1,\ldots,n \; .$$

Proof. One uses the fact that any Borel subset is of the form $G \setminus N$, where G is a G_δ set and N is a null set (a set of measure zero).□

Remark 3.3.1. To approximate the numbers $x_{j,n}^k, y_{j,n}^k$, one can make use of Fourier approximate expansion of h with respect to the orthonormal sequence attached to the functions kt^{k-1} via Gram-Schmidt algorithm, also using the values of the moments m_k . Thus one obtains a smooth approximation $\widetilde{h}$ of h, and the intervals of ends $y_{l,n}, x_{l,n}$ are connected components of the open sets approximating from above subsets of the following form, in the sense of the measures of these sets:

$$\left\{ t; \frac{j_l}{2^{p(n)}} \le \widetilde{h}(t) < \frac{j_l + 1}{2^{p(n)}} \right\}.$$

Remark 3.3.2. A similar result to that of Theorem 3.3.1 in several dimensions holds, with the same proof. We state it for the two-dimensional case.

Theorem 3.3.2. *Let* $\left(m_{(j_1,j_2)}\right)_{\substack{1 \le j_1 \le n_1 \\ 1 \le j_2 \le n_2}}$ *be a given family of real numbers, and consider the functions*

$$\psi_{(j_1,j_2)}(t_1,t_2) = j_1 j_2 t_1^{j_1 - 1} t_2^{j_2 - 1}, \quad 1 \le j_p \le n_p, \quad p = 1,2, (t_1,t_2) \in K_2 = [0,b_1] \times [0,b_2] \; .$$

Consider the following statements:

(a) *there exists a Borel function h,* $0 \le h(t_1,t_2) \le 1$, *such that*

$$\iint_{K_2} (\psi_{(j_1,j_2)} \cdot h)(t_1,t_2)dt_1dt_2 = m_{(j_1,j_2)},\ 1 \le j_1 \le n_1,\ 1 \le j_2 \le n_2\ ;$$

(b) *for any family of scalars* $\left(\lambda_{(j_1,j_2)}\right)_{\substack{1\le j_1\le n_1\\ 1\le j_2\le n_2}}$ *, one has*

$$\sum_{\substack{1\le j_1\le n_1\\ 1\le j_2\le n_2}} \lambda_{(j_1,j_2)} \cdot m_{(j_1,j_2)} \le \sum_{\substack{1\le j_1\le n_1\\ 1\le j_2\le n_2}} \lambda_{(j_1,j_2)} b_1^{j_1} b_2^{j_2}\ ;$$

(c) *there exists a Borel subsets* $B_2 \subset K_2$ *such that*

$$\iint_{B_2} \psi_{(j_1,j_2)}(t_1,t_2)dt_1dt_2 = m_{(j_1,j_2)},\ 1 \le j_p \le n_p,\quad p = 1,2\ .$$

Then (b) ⇒(a) ⇔ (c)

Corollary 3.3.2. *If one of the conditions (a), (c) from Theorem 3.3.2 is accomplished, then there exist sequences*

$$y_{1,n} < x_{1,n} < \ldots < y_{l,n} < x_{l,n} < \ldots,$$
$$v_{1,n} < u_{1,n} < \ldots < v_{l,n} < u_{l,n} < \ldots,\ n \in \mathrm{N},$$

such that

$$m_{(k_1,k_2)} = inf_{n\in\mathbb{N}} \left\{ \sum_{l\in\mathbb{N}} \left(x_{l,n}^{k_1} - y_{l,n}^{k_1}\right)\left(u_{l,n}^{k_2} - v_{l,n}^{k_2}\right) \right\}.$$

Proof. The Borel subset B_2 is the joint of a G_δ set and a null set. For an open subset $D_n \supset B_2$, we consider its decomposition into cells used in the construction of Lebesgue measure. If we determine smooth approximations $\widetilde{h}$ *of h* by means of Gram Schmidt algorithm for the functions $\psi_{(j_1,j_2)}$ and the given moments $m_{(j_1,j_2)}$ via Fourier expansion, then the sequences from the present corollary can be determined by means of cell-decomposition of the open subsets which approximate (in measure) the subsets

$$\left\{ (t_1,t_2); \frac{m(l)}{2^{p(n)}} \le \widetilde{h}(t_1,t_2) < \frac{m(l)+1}{2^{p(n)}} \right\}.$$

Of course, this way one obtains approximations of these numbers. □

The above statements solve the truncated moment problems and sketch an algorithm for determining numbers $y_{l,n}, x_{l,n}, v_{l,n}, u_{l,n}$. The next idea is to solve a full moment problem, by means of passing through the limit, based on a weak compactness argument. The following theorem proposes such a construction, thanks to Krein - Milman theorem. We state it firstly in the one dimensional case, although the several dimensional case follows using similar arguments.

Theorem 3.3.3. *With the notations from Theorem 3.3.1, let* $(m_k)_{k\ge1}$ *be a sequence of real numbers. Consider the following statements:*

(a) *there exists a Borel function h such that*

$$0 \le h(t) \le 1 \quad a.e.,\ m_k = k\int_a^b t^{k-1}h(t)dt,\quad k \in \mathrm{N} \setminus \{0\}\ ;$$

(b) *for any natural number* $n \ge 1$ *, and any* $\varepsilon > 0$ *, there exist nonnegative scalars* β_j *,* $j = 1,\ldots,n$ *, and sequences:*

$y_{j,1} < x_{j,1} < \ldots < y_{j,l} < x_{j,l} < \ldots,\ \ j = 1,\ldots,n$ *such that*

$$1 - \varepsilon \le \sum_{j=1}^{n} \beta_j \le 1,\ \ m_k = \lim_n \left(\sum_{j=1}^{n} \beta_j \left(\sum_l \left(x_{j,l}^k - y_{j,l}^k \right) \right) \right),\ \ k \in \mathrm{N} \setminus \{0\};$$

(c) *for any* $n \in \mathrm{N} \setminus \{0\}$ *, there exists a Borel subset* B_n *such that*

$$m_k = k\int_{B_n} t^{k-1}dt, \quad k = 1,\dots,n.$$

(d) *for any natural* $n \geq 1$ *and any* $\{\lambda_1,\dots,\lambda_n\} \subset R$ *the following relation holds true:*

$$\sum_{k=1}^{n} \lambda_k m_k \leq \sum_{k=1}^{n} \lambda_k (b^k - a^k).$$

Then(d) $\Rightarrow$ (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

Proof. *(*a) $\Rightarrow$(b). One applies Krein - Milman Theorem for the weakly compact subset formed by intersecting the unit ball of $L^\infty([0,b])$ with the positive cone of the same space. Then we must have:

$$h = \lim_n \left(\sum_{j=1}^{n} \alpha_j \chi_{B_j} \right) = \lim_n \left(\sum_{j=1}^{n} \beta_j \chi_{D_j} \right), \quad \alpha_j \geq 0, \quad \sum_{j=1}^{n} \alpha_j = 1,$$

$$\beta_j = \frac{m(B_j)}{m(D_j)} \cdot \alpha_j \in [(1-\varepsilon)\alpha_j, \alpha_j], \quad B_j \subset D_j, \quad j = 1,\dots,n, D_j$$

are suitable chosen open subsets, where the limit is in the weak topology on L^∞, with respect to the dual pair (L^1, L^∞); *m* is the Lebesgue measure. This leads to:

$$m_k = k\int_a^b t^{k-1}h(t)dt = \lim_n \left(\sum_{j=1}^{n} \beta_j \int_{D_j} kt^{k-1}dt \right).$$

Since each open subset D_j has an at most countable decomposition

$$D_j = \bigcup_l (y_{j,l}, x_{j,l}),$$

the conclusion (b) follows. For the converse implication, observe that each step function

$$h_n := \sum_{j=1}^{n} \beta_j \chi_{D_j}, \quad D_j = \bigcup_l (y_{j,l}, x_{j,l}), \quad j = 1,\dots,n$$

is an element of the positive part of the unit ball in L^∞ and the latter subset is weakly compact. Let *h* be the $(w)-$ limit of a subsequence of the sequence $(h_n)_n$. Then one obtains

$$m_k = \lim_n \int_a^b h_{m_n}(t)k \cdot t^{k-1}dt = k\int_a^b h(t) \cdot t^{k-1}dt,$$

by Lebesgue dominated convergence theorem. Hence (b)$\Rightarrow$ (a) is proved. The implication (a) $\Rightarrow$ (c) follows from (a) $\Rightarrow$ (c) of Theorem 3.3.1, since a solution of the full moment problem is a solution of all truncated moment problems. It remains to prove that (d) $\Rightarrow$ (a). This is a consequence of the implication (b) $\Rightarrow$ (a) of Theorem 3.1.4. If $J_0 \subset N \setminus \{0\}$ is a finite subset and $\{\lambda_j; j \in J_0\} \subset R$,then the following implications hold true:

$$\sum_{j \in J_0} \lambda_j \cdot j \cdot t^{j-1} = \varphi_2 - \varphi_1, \quad \varphi_p \in L_+^1 \Rightarrow$$

$$\sum_{j \in J_0} \lambda_j \cdot (b^j - a^j) \leq \int_a^b \varphi_2 dt = F_2(\varphi_2) - F_1(\varphi_1), \; F_1 := 0 \Rightarrow$$

$$\sum_{j \in J_0} \lambda_j \cdot m_j \leq \int_a^b \varphi_2 dt = F_2(\varphi_2).$$

Application of Theorem 3.1.4 leads to the existence of a linear functional *F*on $L^1([a,b])$,verifying

$$F\left(kt^{k-1}\right)=m_k,\ k\geq 1,\ k\in \mathrm{N},$$

$$0\leq F(\psi)\leq \int_a^b \psi\, dt,\ \forall \psi\in L_+^1([a,b]).$$

Now the conclusion follows by measure theory.□

Remark3.3.3. For the full moment problem, the following algorithm holds in determining (approximating) $y_{j,l}, x_{j,l}$.

Step 1. (Approximating the function *h*).

Let $(e_n)_{n\geq 1}$ be a Hilbert base constructed by the aid of Gram-Schmidt procedure, applied to the system of linearly independent functions

$$\varphi_n(t)=n\cdot t^{n-1},\quad n\in \mathrm{N}\setminus\{0\}.$$

Then for each fixed natural number $n\geq 1$, one has:

$$e_n=\sum_{j=1}^{n} a_j^{(n)}\varphi_j \Rightarrow \langle h,e_n\rangle=\sum_{j=1}^{n} a_j^{(n)}\langle h,\varphi_j\rangle=\sum_{j=1}^{n} a_j^{(n)} m_j,$$

where the coefficients $a_j^{(n)}$ are known form the Gram-Schmidt procedure. Hence we can determine each Fourier coefficient of *h*, that is we can approximate h in L^2-norm by a sequence of polynomial functions h_n, $n\geq 1$. Then there exists a subsequence $h_{k_n}\to h$ pointwise almost everywhere in $[a,b]$.

Step 2. For each $n\in \mathrm{N}\setminus\{0\}$, the subsets

$$\left\{t;\frac{m_l}{2^p}\leq h_{k_n}(t)<\frac{m_l+1}{2^p}\right\},\quad p,m_l\in \mathrm{N},$$

can be approximated (in measure) by open subsets. The connected components of these open sets have as end points approximations of the unknowns $y_{j,l}, x_{j,l}$. Using a weakly compactness standard argument, we can obtain *h* as the limit of a subsequences of (χ_{B_n}), where B_n are as in assertion (c) of Theorem 3.3.3. Considering a suitable open set $D_{k_n}\supset B_{k_n}$, from (a) one obtains:

$$m_k=\int_{D_{k_n}} k\cdot t^{k-1}dt=\sum_l (x_{n,l}^k-y_{n,l}^k),\quad D_{k_n}=\bigcup_l\ ]y_{n,l},x_{n,l}[\ .$$

This concludes the last remark.
Note that all results of this section can be adapted to the multidimensional moment problem, with similar proofs.

4. ON THE INVARIANCE OF THE UNIT BALL IN L^1 SPACES

This section is exclusively based on the last part of [18]. Let *Y* be an order complete Banach lattice. Let $X=L_\nu^1(A)$, where A,ν are as in Lemma 2..4,*P* being the subspace of polynomials on *A*. Define the set
$S_1^+=\{T\in B(X,Y);\ T(\varphi)\geq 0\ \forall\varphi\in X_+,\ \|T\|\leq 1\}$.

Theorem 4.1.*For a linear operator* $V:P\to Y$, the following statements are equivalent

(a) *V has a linear positive extension* $\widetilde{V}\in S_1^+$;

(b) *There exists* $T\in S_1^+$ *such that* $0\leq V(p)\leq T(p),\ \forall p\in P_+$.

Proof. The implication (a) $\Rightarrow$ (b) is obvious: put $T:=\widetilde{V}\in S_1^+$. Then one has

$$V(p)=\widetilde{V}(p)=T(p), V(p)=\widetilde{V}(p)\geq 0, \forall p\in P_+.$$

To prove the converse, consider a linear positive extension $\overline{V}$ to V, to the subspace of X formed by all functions dominated on A, in absolute value, by a polynomial. The latter space contains both the subspace of polynomials and the subspace of continuous compactly supported functions. The existence of such an extension follows from Theorem 3.1.5 stated above. Let ψ, p_m , $m \in \mathrm{N}$ be as in Lemma2.4, where ψ is continuous and compactly supported. Assume by reduction to absurd that $T(\psi) - \overline{V}(\psi) \notin Y_+$.

Since the positive cone Y_+ is closed and convex, a separation Hahn-Banach result implies the existence of a linear positive functional $y*$ on Y such that

$$y*(T(\psi) - \overline{V}(\psi)) < 0 ,$$

that is

$$y*(T(\psi)) < y*(\overline{V}(\psi)) .$$

On the other hand, since all the polynomials p_m are majorizing the nonnegative function ψ , using Fatou's lemma for the linear positive functional $y*\circ\overline{V}$, which can be represented by a positive measure, the following relations hold true

$$y*(\overline{V}(\psi)) \leq \liminf (y*\circ\overline{V})(p_m) = \liminf (y*\circ V)(p_m) \leq \liminf (y*\circ T)(p_m)$$
$$= \lim_m (y*\circ T)(p_m) = y*(T(\psi)).$$

Hence we have been leaded to the contradiction

$$y*(T(\psi)) < y*(T(\psi)) .$$

The conclusion is

$$T(\psi) - \overline{V}(\psi) \in Y_+ ,$$

that is. $\overline{V}(\psi) \leq T(\psi)$ for all nonnegative continuous compactly supported functions ψ . Now let φ an arbitrary continuous compactly supported function. Then by the preceding relations, the following inequalities hold too

$$|\overline{V}(\varphi)| \leq \overline{V}(|\varphi|) \leq T(|\varphi|) .$$

Since the norm on the Banach lattice Y is solid, we infer that

$$\|\overline{V}(\varphi)\| \leq \|T\| \|\varphi\|_1 \leq \|\varphi\|_1 .$$

Hence $\overline{V}$ is linear, positive, continuous and of norm at most one on the dense subspace of X formed by the continuous compactly supported functions. By a standard density argument, it has a unique linear extension $\widetilde{V}$ to the whole space X, of norm at most one. This bounded extension is also positive on X_+ due to the density of positive polynomials in X_+ (Lemma 2.4). This concludes the proof.□

Denote

$$S_1^+(X) := \{T \in B_+(X);\ \ T(\overline{B}_{1,X}) \subseteq \overline{B}_{1,X}\} ,$$

where X is as above, and $\overline{B}_{1,X}$ is the closed unit ball in X. Let $V : P \to X$ be a linear operator.

Corollary 4.1. *The following statements are equivalent*

(a) *V has a linear positive extension* $\widetilde{V} \in S_1^+(X)$ *;*

(b) *there exists* $T \in S_1^+(X)$ *such that* $0 \leq V(p) \leq T(p),\ \forall p \in P_+$.

Proof. Put in Theorem 4.1 $Y = X = L_V^1(A)$. Then X is an order complete vector lattice (in which any order convergent sequence is convergent in the norm topology). To prove (b) $\Rightarrow$ (a), one applies the corresponding implication from theorem 4.1. Observe also that $\|\widetilde{V}\| \leq 1$ if and only if $\overline{V}(\overline{B}_{1,X}) \subseteq \overline{B}_{1,X}$.□

The next goal is to give some characterizations in terms of quadratic forms (when this fact is allowed by the form of positive polynomials in terms of sums of squares).

Corollary 4.2. *Let* $X = L^1_\nu(\mathbb{R})$, *where* ν *is a positive regular M-determinate measure on R (with finite moments of all orders),* $x_j(t) = t^j$, $t \in \mathbb{R}$, $j \in \mathbb{N}$. *Let* $V : P \to X$ *be a linear operator. The following statements are equivalent*

(a) *V has a linear positive extension* $\tilde{V} \in S_1^+(X)$;

(b) there exists $T \in S_1^+(X)$ such that for any finite subset $\{\lambda_j\}_{j \in J_0} \subset \mathbb{R}$, the following relations hold

$$0 \le \sum_{i,j \in J_0} \lambda_i \lambda_j V(x_{i+j}) \le \sum_{i,j \in J_0} \lambda_i \lambda_j T(x_{i+j}).$$

Proof. One applies Corollary 4.1 to $X = L^1_\nu(\mathbb{R})$, when in Theorem 4.1 one takes $n = 1$, $A = \mathbb{R}$, also using the form of positive polynomials on the real line, as being sums of squares of some polynomials with real coefficients.□

Corollary 4.3. *Let* $X = L^1_\nu([0,\infty))$, ν *being a positive regular M- determinate Borel measure on* $\mathbb{R}_+$, *with finite moments of all orders. Let* $x_j(t) = t^j$, $t \in \mathbb{R}_+$, $j \in \mathbb{N}$. *Let* $V : P \to X$ *be a linear operator. The following statements are equivalent*

(a) *V has a linear positive extension* $\tilde{V} \in S_1^+(X)$;

(b) *there exists* $T \in S_1^+(X)$ *such that for any finite subset* $\{\lambda_j\}_{j \in J_0} \subset \mathbb{R}$, *the following relations hold*

$$0 \le \sum_{i,j \in J_0} \lambda_i \lambda_j V(x_{i+j+l}) \le \sum_{i,j \in J_0} \lambda_i \lambda_j T(x_{i+j+l}), \quad l \in \{0,1\}.$$

Proof. The proof is similar to that of Corollary 4.2, also using the form of positive polynomials on $\mathbb{R}_+$.□

As it is well known, in several dimensions, there are positive polynomials which are not sums of squares cf. [2]. However, using approximation results (Lemma 2.5), the connection with tensor products of positive polynomials in each separate variable holds true. Let $X := L^1_\nu(\mathbb{R}^n)$ where ν is as in Lemma 2.5,

$$x_j(t_1,\dots,t_n) = t_1^{j_1} \dots t_n^{j_n}, \quad j = (j_1,\dots,j_n) \in \mathbb{N}^n, \ (t_1,\dots,t_n) \in \mathbb{R}^n .$$

Let $V : P \to X$ be a linear operator, where P is the subspace of polynomials in n real variables, with real coefficients.

Theorem 4.2. *The following statements are equivalent*

(a) *V has a linear positive extension* $\tilde{V} \in S_1^+(X)$;

(b) $0 \le V(p)$, $\forall p \in P_+$ *and there exists* $T \in S_1^+(X)$ *such that for any finite subset* $J_k \subset N$, $k = 1,\dots,n$ *and any*

$$\{\lambda_{j_k}\}_{j_k \in J_k} \subset \mathbb{R}, \quad k = 1,\dots,n,$$

one has

$$\sum_{i_1, j_1 \in J_1} \left(\cdots \left(\sum_{i_n, j_n \in J_n} \lambda_{i_1} \lambda_{j_1} \dots \lambda_{i_n} \lambda_{j_n} V(x_{i_1+j_1,\dots,i_n+j_n}) \right) \cdots \right) \le$$
$$\sum_{i_1, j_1 \in J_1} \left(\cdots \left(\sum_{i_n, j_n \in J_n} \lambda_{i_1} \lambda_{j_1} \dots \lambda_{i_n} \lambda_{j_n} T(x_{i_1+j_1,\dots,i_n+j_n}) \right) \cdots \right).$$

Corollary 4.4. *Let* $X = L^1_\nu(\mathbb{R}^n)$ *and Y be a Banach lattice. Assume that T is a linear bounded operator fromX into Y. The following statements are equivalent*

(a) *T is a positive linear operator from X into Y;*

(b) for any finite subsets $J_k \subset \mathrm{N}$, $k = 1,\ldots,n$ *and any*

$$\{\lambda_{j_k}\}_{j_k \in J_k} \subset \mathrm{R}, \quad k = 1,\ldots,n,$$

the following relation holds

$$0 \le \sum_{i_1, j_1 \in J_1} \left(\cdots \left(\sum_{i_n, j_n \in J_n} \lambda_{i_1} \lambda_{j_1} \cdots \lambda_{i_n} \lambda_{j_n} T(x_{i_1 + j_1, \ldots, i_n + j_n}) \right) \cdots \right).$$

Proof. Notice that (b) says that T is positive on the convex cone generated by special positive polynomials mentioned in lemma 2. Consequently, (a) $\Rightarrow$ (b) is obvious. In order to prove the converse, observe that any nonnegative element of X can be approximated by nonnegative continuous compactly supported functions. Such functions can be approximated by sums of tensor products of positive polynomials in each separate variable (Lemma 2.5). The conclusion is that any nonnegative function from X can be approximated in $X = L_\nu^1(\mathrm{R}^n)$ by sums of tensor products of squares of polynomials in each separate variable. But on such special polynomials, T admits nonnegative values, following the condition (b). Now the desired conclusion is a consequence of the continuity of T. This concludes the proof. □

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