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RESEARCH ARTICLE

REPRODUCING GROUPS ON ANALYTIC FEATURESFOR THE METAPLECTIC REPRESENTATION

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ABSTRACT

We introduce the notation of admissible subgroup H of $G = \mathbb{R}^{(1+\varepsilon)} \rtimes Sp(1+\varepsilon,\mathbb{R})$ relative to the (extended) metapletic representation μ_e via the wigner distribution. Under mild additional assumptions, it is shown to be equivalent to the fact that the identity $\sum_j f_j = \int_H \sum_j \langle f_j, \mu_e(h_j)\phi_j\rangle\, \mu_e(h_j)\phi_j dh_j$ holds (weakly) for all $f_j \in L^2(\mathbb{R}^{(1+\varepsilon)})$. They used this equivalence to exhibit classes of admissible subgroups of $Sp(2,\mathbb{R})$ by E. Cordero, F. Damai, K. Nowak, and A. Tabacco [20]. We also eastablish some connections with wavelet theory, i.e., with curvelet and contourlet frames.

INTRODUCTION

The study of reproducing formulae for functions in $L^2(\mathbb{R}^{(1+\varepsilon)})$ has attracted, in physics (Ali *et al.,* 2000), group representations (Dixmier and Les, 1996) and applied mathematics, both in Gabor analysis (Grochenig *et al.,* 2001) and in wavelet theory (Candes *et al.,* 2001; Do and Vetterli; Laugesen *et al.,* 2002). In a very general and abstract sense, they can all be recast in a formula of the series type.

$$\int_H \sum_j \langle f_j, (\phi_j)_{h_j}\rangle (\phi_j)_{h_j} dh_j, \quad f_j \in \mathcal{H} \tag{1}$$

where $\mathcal{H}$ is a Hilbert space and $h_j \longmapsto (\phi_j)_{h_j}$ is an $\mathcal{H}$-valued measurable function on somemeasure space (H, $(1+\varepsilon)h_j$).

The cases of greatest interest concern Hilbert spaces of functions, while the measure space H serves as parameter space. Thus, H takes into account the particular kind of analysis and synthesis processes that a formula like (1), known as reproducing formula, is meant to describe. We are mostly interested in the case in which His a Lie group with left Haar measure dh_j, $\phi_j \in L^2(\mathbb{R}^{(1+\varepsilon)})$ is fixed and $h_j \longmapsto (\phi_j)_{h_j}$ is an $L^2(\mathbb{R}^{(1+\varepsilon)})$valued unitary representation of H. This rich structure often provides both a very efficient tool for computations and a means for finding new reproducing formulae, specially when H is chosen among the subgroups of some classical group of linear symmetries. A class of groups that has been widely studied is the class of semi direct products H = $\mathbb{R}^{(1+\varepsilon)} \rtimes$ D, whereD is a closed matrix group (the so-called dilation group). admitted a natural unitaryrepresentation on $L^2(\mathbb{R}^{(1+\varepsilon)})$, the main ingredient for the construction of a wavelet transform by E. Cordero, F. Damai, K. Nowak, and A. Tabacco. Initially, only irreducible square-integrable representations were considered [2, 12], but it soon became clear that nonirreducible representations (Grochenig *et al.,* 1992; Mallat and Zhong, 1992; Fuhr and Mayer, 2002) are of relevance as well. The authors of [18] have proved a characterization of those dilation groups D which give rise to a reproducing formula (1). They introduce a notion of admissibility,a sufficient condition for a subgroup D of GL($\mathbb{R}$, $1+\varepsilon$) to admit sequence of windows$\phi_j \in L^2(\mathbb{R}^{(1+\varepsilon)})$ such that (1) works for all$f_j \in L^2(\mathbb{R}^{(1+\varepsilon)})$.

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A dilation group D is admissible if there exists a Borel measurable $(1+\varepsilon) \in L^1(\mathbb{R}^{(1+\varepsilon)})$ such that $\varepsilon > -1$ and

$$\int_D \sum_j (1+\varepsilon)\left(\sum_r x_r^{\mathrm{j}}\,\left({}^{(1+\varepsilon)}a\right)\right) da = 1, \quad for\ a.e.\ \sum_r x_r^{\mathrm{j}} \in \mathbb{R}^{(1+\varepsilon)} \tag{2}$$

where $\left({}^{(1+\varepsilon)}a\right)$ is the transpose of the matrix a, $\sum_r x_r^j \longmapsto \sum_r x_r^j \left({}^{(1+\varepsilon)}a\right)$is the right action of a ∈ D, and da is the left Haar measure on D. The above definition is motivated by the analysis of the "a($\sum_r x_r^{\mathrm{j}}$)+ (a + ε)" group. In that case, any admissible wavelet ψ (in the usual Calderón sense) gives a function (1 + ε) = $|\hat{\psi}|^2$ for which formula (2) holds.

We work in a somewhat different setting. First, the Lie group H in (1) is a subgroup of the semidirect product G = $\mathbb{R}^{(1+\varepsilon)} \rtimes$ Sp$(1+\varepsilon,\ \mathbb{R})$ of the Heisenberg group and the symplectic group. Secondly, the representation $h_j \longmapsto (\phi_j)_{h_j}$ arises from the restriction to H of the reducible(extended) metaplectic representation μ_e of G as applied to a fixed and suitable sequence of window functions $\phi_j \in L^1(\mathbb{R}^{(1+\varepsilon)})$. A group H for which there exists the sequence of windows ϕ_j such that (1) holds is said to be reproducing. A complete classification of reproducing subgroups in the case $\varepsilon = 0$ is given in [8], but for the case $\varepsilon > 1$, the groups we treat here are the only known examples. Although, the setups $\mathbb{R}^{(1+\varepsilon)} \rtimes$ D and $\mathbb{H}^{(1+\varepsilon)} \rtimes$Sp$(1+\varepsilon, \mathbb{R})$ are quite different in spirit, there is a crucial conceptual link between them. The point is that both are intimately related to the geometry of affine actions on Euclidean space. Indeed, one of the most important features of μ_e is that it may be realized by affine actions on $\mathbb{R}^{2(1+\varepsilon)}$by means of the Wigner distribution. The referred to (Claasen *et al.,* 1980; Folland, 1998; Grochenig, 2001) for a thorough discussion of this basic construct in time-frequency analysis.

The cross-Wigner distributions $W^j_{f_j,\mathrm{g}^j}$of f_j, $\mathrm{g}^{\mathrm{j}} \in L^2(\mathbb{R}^{(1+\varepsilon)})$ are

$$\sum_j W^j_{f_j,\mathrm{g}^j}\left(\sum_r x_r^j, \sum_r \xi_r^j\right) = \int \sum_j e^{-2\pi i \langle \sum_r \xi_r^j, y^j\rangle} f_j\left(\sum_r x_r^j + \frac{y^j}{2}\right) \overline{\mathrm{g}^{\mathrm{j}}\left(\mathrm{y}^{\mathrm{j}} + \frac{\mathrm{y}^{\mathrm{j}}}{2}\right)}\, dy^{\mathrm{j}} \tag{3}$$

The quadratic expression $W^j_{f_j} := W^j_{f_j,f_j}$is usually called the Wigner distributions of f_j. The crucial properties of W^jalluded to above is that it intertwines μ_e and the affine action on $\mathbb{R}^{2(1+\varepsilon)}$. In other words:

$$W^j_{\mu_e(\mathrm{g}^{\mathrm{j}})\phi_j}\left(\sum_r x_r^j, \sum_r \xi^{j\,\mathrm{j}}_{\ r}\right) = W^j_{\phi_j}\left((\mathrm{g}^{\mathrm{j}})^{-1} \cdot \left(\sum_r x_r^j, \sum_r \xi_r^j\right)\right),\ \mathrm{g}^{\mathrm{j}} \in \mathrm{G},$$

Where $\mathrm{g}^{\mathrm{j}} \cdot (\sum_r x^{j\,\mathrm{j}}_{\ r}, \sum_r \xi^{j\,\mathrm{j}}_{\ r})$ is the natural affine action of G on phase space. Actually, since there producing formula is insensitive to phase factors, i.e., to the action of the center of $\mathbb{H}^{(1+\varepsilon)}$, the group G is truly$\mathbb{R}^{2(1+\varepsilon)} \rtimes$ Sp$(1+\varepsilon,\ \mathbb{R})$, whose affine action on $\mathbb{R}^{2(1+\varepsilon)}$is rather obvious. This is why in our Definition 8 the Wigner distribution plays the same role a$(1+\varepsilon)$plays in (2). Thus, we call admissible a connected Lie subgroup H ⊂ G if there exists$\phi_j \in L^2(\mathbb{R}^{(1+\varepsilon)})$ such that

$$\int_H \sum_j W^j_{\phi_j}\left(h_j^{-1}.\left(\sum_r x_r^j, \sum_r \xi_r^j\right)\right) dh_j = 1, \quad for\ a.e.\left(\sum_r x_r^j, \sum_r \xi_r^j\right) \in \mathbb{R}^{2(1+\varepsilon)}$$

We exhibit another reproducing subgroup of Sp(2, ℝ), that we denote TDS(2), which is a covering of the similitude group of the plane SIM(2). We then show that our theory, for both TDS(2) and SIM(2), parallels the theory developed in the context of two-dimensional wavelets. The groups TDS(2) and SIM(2) are the forerunners of the curvelet and contourlet frames, nowadays heavily employed in the context of signal processing [4, 10]. In particular, curvelets are actively investigated from the point of view of statistical estimation, sparsity of the representation and rate of approximation. The approach starts from the whole time-frequency plane $\mathbb{R}^{2(1+\varepsilon)}$, instead of looking at either time or frequency, as is typical in the philosophy of the setting $\mathbb{R}^{(1+\varepsilon)} \rtimes$ D. This justifies the useof the Wigner distribution and its time-frequency properties. We show that a class of groups, parametrized by $\beta \in \mathbb{R}$ and including SIM(2) when ε = -β, is reproducing. This time, however, our proof is direct, namely we show (1) without using Theorem 7.

Preliminaries and Notation

The symplectic group is defined by

$$\mathrm{Sp}(1+\varepsilon, \mathbb{R}) = \left\{\mathrm{g}^{\mathrm{j}} \in \mathrm{GL}(2(1+\varepsilon), \mathbb{R}) \colon {}^{(1+\varepsilon)}\mathrm{g}^{\mathrm{j}}(\,\mathrm{J}\,)\,\mathrm{g}^{\mathrm{j}} = \mathrm{J}\right\}$$

where

$$J = \begin{bmatrix} 0 & I_{(1+\varepsilon)} \\ -I_{(1+\varepsilon)} & 0 \end{bmatrix}$$

is the standard symplectic form

$\omega(\sum_r x_r^j, y^j) = {}^{(1+\varepsilon)}\sum_r x_r^j\,(\mathrm{J})y^j \sum_r x_r^j, y^j \in \mathbb{R}^{2(1+\varepsilon)}$. (4)

The metaplectic representation μ of (the two-sheeted cover of) the symplectic group arises as intertwining operator between the standard Schrödinger representation ρ of the Heisenberg group $\mathbb{H}^{(1+\varepsilon)}$and the representation that is obtained from it by composing ρ with the action of Sp($1+\varepsilon$,ℝ) by automorphisms on $\mathbb{H}^{(1+\varepsilon)}$ (see, e.g., (Folland, 1998)). We briefly review its construction. The Heisenberg group $\mathbb{R}^{(1+\varepsilon)}$is the group obtained by defining on $\mathbb{R}^{2\varepsilon+3}$ the product

$$(z^j, 1+\varepsilon)\cdot\left(\acute{z}^j\cdot\left(\frac{1+\varepsilon}{\varepsilon}\right)\right) = \left(z^j + \acute{z}^j, \left(\frac{\varepsilon^2+\varepsilon+1}{\varepsilon}\right) - \frac{1}{2}\omega\left(z^j, \acute{z}^j\right)\right)$$

where ω stands for the standard symplectic form in $\mathbb{R}^{2(1+\varepsilon)}$ given in (4). We denote the translation and modulation operators on $L^2(\mathbb{R}^{(1+\varepsilon)})$ by

$T_{(\sum_r x_r^j)} f_j\,(1+\varepsilon) = f_j\,((1+\varepsilon) - \sum_r x_r^j)$ and $M_{(\sum_r \xi_r^j)} f_j\,(1+\varepsilon) = e^{2\pi i\langle \sum_r \xi_r^j, 1+\varepsilon\rangle} f_j\,(1+\varepsilon)$.

The Schrödinger representation of the group $\mathbb{H}^{(1+\varepsilon)}$on $L^2(\mathbb{R}^{(1+\varepsilon)})$ is then defined by

$$\rho\left(\sum_r x_r^j, \sum_r \xi_r^j, 1+\varepsilon\right) f_j(y^j) = e^{2\pi i(1+\varepsilon)} e^{\pi i\langle \sum_r x_r^j, \sum_r \xi_r^{j}\rangle} e^{2\pi i\langle \sum_r \xi_r^j, y^j\rangle} f_j\left(y^j - \sum_r x_r^j\right)$$

$$= e^{2\pi i(1+\varepsilon)} e^{\pi i\langle \sum_r x_r^j, \sum_r \xi_r^j\rangle} T_{(\sum_r x_r^j)} M_{(\sum_r \xi_r^j)} f_j(1+\varepsilon)$$

where we write $z^j = (\sum_r x_r^j, \sum_r \xi_r^j)$ when we separate space components (that are$\sum_r x_r^j$) from frequency components (that are $\sum_r \xi_r^j$) in points z^j in phase space $\mathbb{R}^{2(1+\varepsilon)}$. The symplectic group acts on $\mathbb{H}^{(1+\varepsilon)}$via automorphisms that leave the center $\{(0, 1+\varepsilon) : (1+\varepsilon) \in \mathbb{R}\} \in \mathbb{H} \simeq \mathbb{R}$ of$\mathbb{H}^{(1+\varepsilon)}$pointwise fixed:

$A_j \cdot (z^j, 1+\varepsilon) = (A_j z^j, 1+\varepsilon)$.

Therefore, for all fixed $A_j \in$ Sp($1+\varepsilon$, ℝ) there is a representation

$\rho_{A_j} : \mathbb{H}^{(1+\varepsilon)} \to \mathcal{U}\,(L^2(\mathbb{R}^{(1+\varepsilon)})), (z^j, 1+\varepsilon) \mapsto \rho\,(A_j \cdot (z^j, 1+\varepsilon))$

whose restriction to the center is a multiple of the identity. By the Stone-von Neumann theorem, $\rho_{A_j}$are equivalent to ρ. That is, there exists an intertwining unitary operator $\mu(A_j) \in \mathcal{U}(L^2(\mathbb{R}^{(1+\varepsilon)}))$ such that $\rho_{A_j}(z^j, 1+\varepsilon) = \mu(A_j) \circ \rho(z^j, 1+\varepsilon) \circ \mu A_j^{-1}$, for all $(z^j, 1+\varepsilon) \in \mathbb{H}^{(1+\varepsilon)}$. By Schur'slemma, μ is determined up to a phase factors $e^{is^j}, s^j \in \mathbb{R}$. It turns out that the phase ambiguity is really a sign, so that μ lifts to a representation of the (double cover of the) symplectic group. It is the famous metaplectic or Shale-Weil representation. The representations ρ and μ can be combined and give rise to the extended metaplectic representation of the group G = $\mathbb{H}^{(1+\varepsilon)} \rtimes$ Sp($1+\varepsilon$, ℝ), the semidirect product of $\mathbb{H}^{(1+\varepsilon)}$ and

Sp($1+\varepsilon$, ℝ).The group law on G are

$$\left((z^j, 1+\varepsilon), A_j\right)\cdot\left(\left(\acute{z}^j, \left(\frac{1+\varepsilon}{\varepsilon}\right)\right), \acute{A}_j\right) = \left((z^j, 1+\varepsilon)\ \cdot\left(A_j\acute{z}^j, \left(\frac{1+\varepsilon}{\varepsilon}\right)\right), A_j\acute{A}_j\right)$$

and the extended metaplectic representation μ_e of G are

$\mu_e\left((z^j, 1+\varepsilon), A_j\right) = \rho(z^j, 1+\varepsilon) \circ \mu(A_j)$.

A slight simplification in our formalism comes from the observation that the reproducing formula (1) is insensitive to phase factors: If we replaces $\mu_e\,(h_j)\phi_j := (\phi_j)_{h_j}$ with $e^{is^j}\mu_e\,(h_j)\phi_j$the formula is unchanged, for all $s^j \in \mathbb{R}$. The role of the center of the Heisenberg group is thus irrelevant, so that the "true" group under consideration is $\mathbb{R}^{2(1+\varepsilon)} \rtimes$ Sp($1+\varepsilon$, ℝ), which we denote again by
G. Thus, G acts naturally by affine transformations on phase spaces, namely

$g^j \cdot (\sum_r x_r^j, \sum_r \xi_r^j) = ((1+\varepsilon, 1+\varepsilon), A_j) \cdot (\sum_r x_r^j, \sum_r \xi_r^j) = A_j({}^{(1+\varepsilon)}(\sum_r x_r^j, \sum_r \xi_r^j)) + {}^{(1+\varepsilon)}(1+\varepsilon, 1+\varepsilon)$ (5)

For elements of Sp($1+\varepsilon$, ℝ) in special form, the metaplectic representation can be computed explicitly in a simple way. For

$f_j \in L^2(\mathbb{R}^{(1+\varepsilon)})$, we have

$$\mu\left(\begin{bmatrix} A_j & 0 \\ 0 & {}^{(1+\varepsilon)}A_j^{-1} \end{bmatrix}\right) f_j(\textstyle\sum_r x_r^j) = det\,(A_j)^{-1/2} f_j(A_j^{-1}(\sum_r x_r^j)) \tag{6}$$

$$\mu\left(\begin{bmatrix} 1 & 0 \\ C & 1 \end{bmatrix}\right) f_j(\textstyle\sum_r x_r^j) = \pm e^{-i\pi\langle C \sum_r x_r^j, \sum_r x_r^j\rangle} f_j(\sum_r x_r^j) \tag{7}$$

$$\mu(\mathrm{J}) = i^{(1+\varepsilon)/2}\mathcal{F}^{-1} \tag{8}$$

where $\mathcal{F}$ denotes the Fourier transform

$$\textstyle\sum_j \mathcal{F} f_j(\sum_r \xi_r^j) = \int_{\mathbb{R}^{(1+\varepsilon)}} \sum_j f_j(\sum_r x_r^j) e^{-2\pi i\langle \sum_r x_r^j, \sum_r \xi_r^j\rangle}\, d(\sum_r x_r^j), \quad f_j \in L^1(\mathbb{R}^{(1+\varepsilon)}) \cap L^2(\mathbb{R}^{(1+\varepsilon)})$$

In the above formula and elsewhere, $\langle \sum_r x_r^j, \sum_r \xi_r^j \rangle$denotes the inner products of $\sum_r x_r^j, \sum_r \xi_r^j \in \mathbb{R}^{(1+\varepsilon)}$. Similarly, for f_j, $g^j \in L^2(\mathbb{R}^{(1+\varepsilon)})$, $\langle f_j, g^j\rangle$ will denote their inner product in $L^2(\mathbb{R}^{(1+\varepsilon)})$. Other notation is as follows. We put$\dot{\mathbb{R}} = \mathbb{R} \setminus \{0\}$, $\mathbb{R}_\pm = (0, \pm\infty)$. For $0 \le \varepsilon \le \infty$, $\|\cdot\|_{1+\varepsilon}$ stands for the$L^{1+\varepsilon}$ -norm of measurable functions on $\mathbb{R}^{(1+\varepsilon)}$ with respect to Lebesgue measure. The left Haar measure of a group H will be written dh_j and we always assume that the Haar measure of a compact group is normalized so that the total mass is one.

The Reproducing Condition

Definition 1. We say that a connected Lie subgroup H of G = $\mathbb{R}^{2(1+\varepsilon)} \rtimes \mathrm{Sp}(1+\varepsilon, \mathbb{R})$ is areproducing group for μ_e if there exists the sequence of functions $\phi_j \in L^2(\mathbb{R}^{(1+\varepsilon)})$ such that

$$\textstyle\sum_j f_j = \int_H \sum_j \langle f_j, \mu_e(h_j)\phi_j\rangle\, \mu_e(h_j)\phi_j dh_j, \quad for\ all\ f_j \in L^2(\mathbb{R}^{(1+\varepsilon)}) \tag{9}$$

All $\phi_j \in L^2(\mathbb{R}^{(1+\varepsilon)})$ for which (9) holds is called reproducing sequence of function.

Notice that we do require formula (9) to hold for all functions in $L^2(\mathbb{R}^{(1+\varepsilon)})$for the same sequence of windows ϕ_j, but we do not require the restriction of μ_e to H to be irreducible. Equivalently, formula (9) can be written in term of the L^2-norm of f_j

$$\textstyle\sum_j \|f_j\|^2 = \int_H \sum_j |f_j, \mu_e(h_j)\phi_j|^2\, dh_j, for\ ball\ f_j \in L^2(\mathbb{R}^{(1+\varepsilon)}) \tag{10}$$

The Wigner distribution and some useful properties

We collect some well-known properties of the Wigner distribution and then we establish Lemma 4 and 5 and 6. For the proof of Proposition 2 see (Folland, 1998; Grochenig, 2001), whereas Lemma 3 is from (Grochenig, 2001). Recall that the cross-Wigner distribution is defined, for f_j, $g^j \in L^2(\mathbb{R}^{(1+\varepsilon)})$, by (3).

Proposition 2. The Wigner distribution of f_j, $g^j \in L^2(\mathbb{R}^{(1+\varepsilon)})$ satisfies:

(i) $W^j_{f_j, g^j}$ are uniformly continuous on $\mathbb{R}^{2(1+\varepsilon)}$, and $\sum_j \left\|W^j_{f_j,g^j}\right\|_\infty \le 2^{(1+\varepsilon)} \|\sum_j f_j\|_2 \|g^j\|_2$

(ii) $W^j_{f_j,g^j} = \overline{W^j_{f_j,g^j}}$ in particular, $W^j_{f_j}$ are real-valued.

(iii) $Moyals$ identity : $\langle W^j_{f_j}, W^j_{g^j}\rangle_{L^2(\mathbb{R}^{2(1+\varepsilon)})} = \langle f_j, g^j\rangle_{L^2(\mathbb{R}^{2(1+\varepsilon)})} \overline{\langle f_j, g^j\rangle}_{L^2(\mathbb{R}^{(1+\varepsilon)})}$

(iv) $If\ f_j, g^j \in S(\mathbb{R}^{(1+\varepsilon)})$, then $W^j_{f_j,g^j} \in \mathcal{S}(\mathbb{R}^{2(1+\varepsilon)})$

(v) $If\ f_j \in L^1(\mathbb{R}^{(1+\varepsilon)}) \cap L^2(\mathbb{R}^{(1+\varepsilon)})$

then $\sum_j \|f_j\|_2^2 = \int_{\mathbb{R}^{2(1+\varepsilon)}} \sum_j W^j_{f_j}(\sum_r x_r^j, \sum_r \xi_r^j) d(\sum_r x_r^j)\, d(\sum_r \xi_r^j)$

An alternative description of $W^j_{f_j,g^j}$is provided by the lemma below (see e.g., [20]).

Lemma 3. Let $\mathcal{T}_{s^j}(1+\varepsilon)(\sum_r x_r^j, 1+\varepsilon) = (1+\varepsilon)(\sum_r x_r^j + \frac{(1+\varepsilon)}{2}, \sum_r x_r^j - \frac{(1+\varepsilon)}{2})$

Be symmetric coordinate transform and

$$\mathcal{F}_2(1+\varepsilon)\textstyle\sum_j \left(\sum_r x^j{}_r\ , \sum_r \xi_r^j\right) = \int_{\mathbb{R}^{(1+\varepsilon)}} (1+\varepsilon) \sum_j (\sum_r x_r^j, 1+\varepsilon)\, e^{-2\pi i\left(\sum_r x_r^j, 1+\varepsilon\right)} d(1+\varepsilon)$$

be the Fourier transform in the second variable.
Then
$W^j_{f_j,g^j} = \mathcal{F}_2\mathcal{T}_{s^j}\ (f_j \otimes \overline{g^j})$ (11)

We use Lemma 3 to prove the following density result.

Lemma 4. If $R(\sum_r x^j_r, \sum_r \xi^j_r)$ are real, slowly increasing measurable function on $\mathbb{R}^{2(1+\varepsilon)}$ such that

$$\int_{\mathbb{R}^{2(1+\varepsilon)}} \sum_j R\left(\sum_r x^j_r, \sum_r \xi^j_r\right) W^j_{f_j}\left(\sum_r x^j_r, \sum_r \xi^j_r\right) d\left(\sum_r x^j_r\right) d\left(\sum_r \xi^j_r\right),\ for\ all\ f_j \in S(\mathbb{R}^{(1+\varepsilon)})$$

then $R\ (\sum_r x^j_r, \sum_r \xi^j_r) = 0$ for a.e. $(\sum_r x^j_r, \sum_r \xi^j_r) \in \mathbb{R}^{2(1+\varepsilon)}$.

Proof. By Lemma 3 it follows that V:= span $\{W^j_{f_j,g^j} f_j, g^j \in \mathcal{S}(\mathbb{R}^{(1+\varepsilon)})\}$ are dense in $\mathcal{S}(\mathbb{R}^{(1+\varepsilon)})$.For $f_j, g^j \in S(\mathbb{R}^{(1+\varepsilon)})$, a straightforward computation gives

$$W^j_{f_j+ig^j} = W_{f_j} + W_{g^j} + 2\mathcal{R}eW^j_{f_j,g^j},\ W^j_{f_j+ig^j} = W^j_{f_j} + W^j_{g^j} + 2ImW^j_{f_j,g^j}$$

and the assumption implies $\langle R, \mathcal{R}eW^j_{f_j,g^j}\rangle = 0$ and $\langle R, ImW^j_{f_j,g^j}\rangle = 0$. Since R is real, these two identities are equivalent to $\langle R, W^j_{f_j,g^j}\rangle = 0$. The conclusion follows from the density of V, because for every $(1+\varepsilon) \in \mathcal{S}(\mathbb{R}^{2(1+\varepsilon)})$ the functional

$$(1+\varepsilon) \longmapsto \int_{\mathbb{R}^{2(1+\varepsilon)}} \sum_j R\left(\sum_r x^j_r, \sum_r \xi^j_r\right) \overline{(1+\varepsilon)\ \left(\sum_r x^j_r, \sum_r \xi^j_r\right)} d\left(\sum_r x^j_r\right) d\left(\sum_r \xi^j_r\right)$$

are atempered distributions and we have

$$\int_{\mathbb{R}^{2(1+\varepsilon)}} \sum_j R\left(\sum_r x^j_r, \sum_r \xi^j_r\right) \overline{(1+\varepsilon)\left(\sum_r x^j_r, \sum_r \xi^j_r\right)} d\left(\sum_r x^j_r\right) d\left(\sum_r \xi^j_r\right) = \langle R, (1+\varepsilon)\rangle$$

$$= lim_{n\to\infty} \langle R, \sum_{k=0}^n c_k W^j_{(f_j)_k,(g^j)_k}\rangle$$

$$= lim_{n\to\infty} \langle R, \sum_{k=0}^n c_k W^j_{(f_j)_k,(g^j)_k}\rangle = lim_{n\to\infty} \sum_{k=0}^n \bar{c}_k \langle R, W^j_{(f_j)_k,(g^j)_k}\rangle = 0$$

so that $R(\sum_r x^j_r, \sum_r \xi^j_r) = 0$, for a.e. $(\sum_r x^j_r, \sum_r \xi^j_r) \in \mathbb{R}^{2(1+\varepsilon)}$ and the proof is complete.

Lemma 5. Let $(\phi_j)_0, (\phi_j)_1 \in L^2(\mathbb{R}^{(1+\varepsilon)})$ and define $\phi_j := (\phi_j)_0 \otimes (\phi_j)_1 \in L^2(\mathbb{R}^{2(1+\varepsilon)})$. Then

$$W^j_{\phi_j}((z^j)_1, (z^j)_2), (\zeta^j)_1, (\zeta^j)_2) = W^j_{(\phi_j)_0}\ (((\zeta^j)_1, (\zeta^j)_2) W^j_{(\phi_j)_1}\ (((\zeta^j)_1, (\zeta^j)_2), \quad (12)$$

where the variables $(z^j)_1, (z^j)_2, (\zeta^j)_1, (\zeta^j)_2$ are in $\mathbb{R}^{(1+\varepsilon)}$.

Proof. Simply compute the Wigner distribution (3) of $\phi_j := (\phi_j)_0 \otimes (\phi_j)_1$.

We find an admissibility condition that, together with some additional integrability and boundedness properties of $h_j \longmapsto W^j_{\phi_j}(h_j^{-1}\cdot (\sum_r x^j_r, \sum_r \xi^j_r))$ implies that a subgroup Hof

G = $\mathbb{R}^{2(1+\varepsilon)} \rtimes$ Sp(1+ε, $\mathbb{R}$) is reproducing.

Lemma 6. Let $\phi_j \in \mathcal{S}(\mathbb{R}^{(1+\varepsilon)})$, $y^j \in \mathbb{R}^{(1+\varepsilon)}$ and let $\widetilde{\phi}_j$ be the Schwartz functions defined by

$\widetilde{\phi}_j(y^j) = y^j \phi_j(y^j)$. Then,

$$W^j_{\widetilde{\phi}_j}\left(\sum_r x^j_r, \sum_r \xi^j_r\right) = \left(\left(\sum_r x^j_r\right)^2 + \frac{1}{16\pi^2}\ (\partial)^2_{\sum_r \xi^j_r}\right) W^j_{\phi_j}\left(\sum_r x^j_r, \sum_r \xi^j_r\right),\quad \left(\sum_r x^j_r, \sum_r \xi^j_r\right) \quad (13)$$

Proof. We use the definition (3) to compute the wigner distributions of $\widetilde{\phi}_j$. Notice that since $\phi_j \in \mathcal{S}(\mathbb{R}^{(1+\varepsilon)})$ and may interchanges $(\partial)^2_{\sum_r \xi_r^j}$ with the integral sign. Namely,

$$\sum_j W^j_{\phi_j}(\sum_r x_r^j, \sum_r \xi_r^j) = \int_{\mathbb{R}(1+\varepsilon)} \sum_j e^{-2\pi i \langle \sum_r \xi_r^j, y^j\rangle} \left(\sum_r x_r^j + \frac{y^j}{2}\right) \phi_j \left(\sum_r x_r^j + \frac{y^j}{2}\right) \left(\sum_r x_r^j - \frac{y^j}{2}\right) \overline{\phi_j \left(\sum_r x_r^j - \frac{y^j}{2}\right)}\, dy^j$$

$$= \int_{\mathbb{R}(1+\varepsilon)} \sum_j e^{-2\pi i \langle \sum_r \xi_r^j, y^j\rangle} \left((\sum_r x_r^j)^2 - \frac{(y^j)^2}{4}\right) \phi_j \left(\sum_r x_r^j + \frac{y^j}{2}\right) \overline{\phi_j \left(\sum_r x_r^j - \frac{y^j}{2}\right)}\, dy^j$$

$$= \sum_j (\sum_r x_r^j)^2 W^j_{\phi_j}(\sum_r x_r^j, \sum_r \xi_r^j) - \frac{1}{4} \int_{\mathbb{R}(1+\varepsilon)} \sum_j (y^j)^2 e^{-2\pi i \langle \sum_r \xi_r^j, y^j\rangle} \phi_j \left(\sum_r x_r^j + \frac{y^j}{2}\right) \overline{\phi_j \left(\sum_r x_r^j - \frac{y^j}{2}\right)}\, dy^j$$

$$= \sum_j (\sum_r x_r^j)^2 \; W^j_{\phi_j}(\sum_r x_r^j, \sum_r \xi_r^j) -$$

$$\left(\frac{1}{4} \int_{\mathbb{R}(1+\varepsilon)} \sum_j \frac{1}{(2\pi i)^2} (\partial)^2_{\sum_r \xi_r^j} \quad e^{-2\pi i \langle \sum_r \xi_r^j, y^j\rangle} \phi_j \left(\sum_r x_r^j + \frac{y^j}{2}\right) \overline{\phi_j \left(\sum_r x_r^j - \frac{y^j}{2}\right)}\, dy^j\right)$$

$$= \sum_j (\sum_r x_r^j)^2 \; W^j_{\phi_j}\left(\sum_r x_r^j, \sum_r \xi_r^j\right) + \sum_j \frac{1}{16\,\pi^2} (\partial)^2_{\xi^j} \; W^j_{\phi_j}(\sum_r x_r^j, \sum_r \xi_r^j)$$

as desired.

The admissibility condition: We find an admissibility condition that, together with some additional integrability and boundedness properties of $h_j \longmapsto W^j_{\phi_j}\left(h_j^{-1} \cdot (\sum_r x_r^j, \sum_r \xi_r^j)\right)$ implies that a subgroup H of $G = \mathbb{R}^{2(1+\varepsilon)} \rtimes Sp(1+\varepsilon, \mathbb{R})$ is reproducing.

Theorem 7. Suppose that $\phi_j \in L^2(\mathbb{R}^{(1+\varepsilon)})$ are such that the mapping

$$h_j \;\longmapsto\; W^j_{\mu_e(h_j)\phi_j}(\sum_r x_r^j, \sum_r \xi_r^j) = \; W^j_{\phi_j}\,(h_j^{-1} \cdot (\sum_r x_r^j, \sum_r \xi_r^j) \tag{14}$$

is in L^1(H) for a.e. $(\sum_r x_r^j, \sum_r \xi_r^j) \in \mathbb{R}^{2(1+\varepsilon)}$ and

$$\int_H \sum_j \left| \; W^j_{\phi_j}(h_j^{-1} \cdot (\sum_r x_r^j, \sum_r \xi_r^j)) \right| dh_j \leq \widetilde{M}, \; for\ a.e \left(\sum_r x_r^j, \sum_r \xi_r^j\right) \in \mathbb{R}^{2(1+\varepsilon)} \tag{15}$$

Then condition (9) holds for all $f_j \in L^1(\mathbb{R}^{(1+\varepsilon)})$ if and only if the following admissibility conditions are satisfied:

$$\int_H \sum_j \; W^j_{\phi_j}\left(h_j^{-1} \cdot (\sum_r x_r^j, \sum_r \xi_r^j)\right) dh_j = 1 \tag{16}$$

$$for\ a.e\ (\sum_r x_r^j, \sum_r \xi_r^j) \in \mathbb{R}^{(1+\varepsilon)}$$

Proof. It is enough to test the reproducing formula (10) on the Schwartz class. Namely, if we show the mappings $\sum_j f_j \longmapsto \sum_j \langle f_j, \mu_e(h_j)\phi_j\rangle$ are an isometry on $S(\mathbb{R}^{(1+\varepsilon)})$ into L^2(H), the pointwise convergence of the coefficients $\sum_j \langle f_j, \mu_e(h_j)\phi_j\rangle$ guarantees that (10) holds for all $f_j \in L^2(\mathbb{R}^{(1+\varepsilon)})$ as well.

Sufficiency. Assume that (16) is true and take $f_j \in \mathcal{S}(\mathbb{R}^{(1+\varepsilon)})$. By (v) of Proposition (2) its L^2-norm can be computed via its Wigner distributions, that are :

$$\sum_j \|f_j\|_2^2 = \int_{\mathbb{R}^{2(1+\varepsilon)}} \sum_j \; W^j_{f_j}(\sum_r x_r^j, \sum_r \xi_r^j) d(\sum_r x_r^j) d(\sum_r \xi_r^j)$$

$$= \int_{\mathbb{R}^{2(1+\varepsilon)}} \left(\int_H \; \sum_j (h_j^{-1}.(\sum_r x_r^j, \omega) dh_j)\right)$$

$$\times \; (W^j_{f_j}(\sum_r x_r^j, \sum_r \xi_r^j) d(\sum_r x_r^j) d(\sum_r \xi_r^j))$$

$$= \int_H \left(\int_{\mathbb{R}^{2(1+\varepsilon)}} \sum_j \; W^j_{\phi_j}\left(h_j^{-1}.(\sum_r x_r^j, \omega)\right) W^j_{f_j}(\sum_r x_r^j, \sum_r \xi_r^j) d(\sum_r x_r^j) d(\sum_r \xi_r^j)\right) dh_j$$

In the last equality, the integral interchange are justified by Fubini Theorem. Indeed, by (14)and (15) we have

$$\int_{\mathbb{R}^{2(1+\varepsilon)}} \int_H \sum_j \left| \; W^j_{\phi_j}(h_j^{-1}.(\sum_r x_r^j, \omega)) dh_j \right| \left| \; W^j_{f_j}(\sum_r x_r^j, \sum_r \xi_r^j) \right| dh_j\, d(\sum_r x_r^j)\, d(\sum_r \xi_r^j)$$

$$\leq M \int_{\mathbb{R}^{2(1+\varepsilon)}} \sum_j \left| \; W^j_{f_j}(\sum_r x_r^j, \sum_r \xi_r^j) \right| d(\sum_r x_r^j)\, d(\sum_r \xi_r^j) < \infty$$

Further, Moyal's identity gives

$$\int_{\mathbb{R}^{2(1+\varepsilon)}} \sum_j \; W^j_{\phi_j}(h_j^{-1}.(\sum_r x^j_r, \omega))\, W^j_{f_j}(\sum_r x^j_r, \sum_r \xi^j_r)\, d(\sum_r x^j_r)\, d(\sum_r \xi^j_r) = \langle W^j_{\mu_e(h_j)}, W^j_{f_j}\rangle = \langle f_j, \mu_e(h_j)\rangle\overline{\langle f_j, \mu_e(h_j)\rangle}$$

hence, the equality Necessity.

$$\sum_j \|f_j\|_2^2 = \int_H \sum_j |\langle f_j, \mu_e(h_j)\phi_j\rangle|^2 \, dh_j\,, \; for\ all\ f_j \in \mathcal{S}(\mathbb{R}^{(1+\varepsilon)})$$

Conversely, assume (9) true and let $f_j$be in $\mathcal{S}(\mathbb{R}^{(1+\varepsilon)})$. Moyal's identityGives

$$\sum_j \|f_j\|_2^2 = \int_{\mathbb{R}^{2(1+\varepsilon)}} \left(\int_H \sum_j \; W^j_{\phi_j}(h_j^{-1}.(\sum_r x^j_r, \omega) dh_j\right) \; W^j_{f_j}(\sum_r x^j_r, \sum_r \xi^j_r) d(\sum_r x^j_r)\, d(\sum_r \xi^j_r) \tag{17}$$

Using again (v) of Proposition 2, equality (17) may be recast as

$$\int_{\mathbb{R}^{2(1+\varepsilon)}} \left(\int_H \sum_j \; W^j_{\phi_j}(h_j^{-1}.(\sum_r x^j_r, \omega) dh_j - 1\right) \; W^j_{f_j}(\sum_r x^j_r, \sum_r \xi^j_r) d(\sum_r x^j_r) d(\sum_r \xi^j_r) = 0$$

The function is real by (ii) of Proposition 2. Hence, (16) follows applying Lemma 4 to it.

$$R(\sum_r x^j_r, \sum_r \xi^j_r) = \int \; \sum_j \; W^j_{\phi_j}(h_j^{-1}.(\sum_r x^j_r, \omega) dh_j - 1)$$

Motivated by Theorem 7, we give the following definition, (see, e.g., [20]).

Definition 8: We say that a connected Lie subgroup H of G = $\mathbb{R}^{2(1+\varepsilon)} \rtimes$Sp(1+$\varepsilon$, ℝ) admissible group for μ_e if there exists the sequence of functions $\phi_j \in L^2(\mathbb{R}^{(1+\varepsilon)})$

$$\int_H \sum_j \; W^j_{\phi_j}(h_j^{-1}.(\sum_r x^j_r, \sum_r \xi^j_r) dh_j) = 1, for\ a.e\ (\sum_r x^j_r, \sum_r \xi^j_r) \in \mathbb{R}^{2(1+\varepsilon)} \tag{18}$$

All $\phi_j \in L^2(\mathbb{R}^{(1+\varepsilon)})$ for which (18) holds are called admissible functions.

It is clear that we now dispose of two different tools for checking whether a subgroup H of G =$\mathbb{R}^{2(1+\varepsilon)} \rtimes$ Sp(1+ε, ℝ) is reproducing or not. Either we find the sequence of window functions ϕ_j for which (9) holds or else we check the admissibility of the subgroup H and use Theorem 7. . We stress that Theorem 7 admits other useful applications (Cordero *et al.,* 2005).

Throughout this section ε = 1. We prove that the 3-dimensional triangular group are reproducing subgroups of Sp(2, ℝ), where

$$TDS(2) = (A_j)_{(1+\varepsilon),\ell^j,y^j} = \begin{bmatrix} (1+\varepsilon)^{-1/2}\mathcal{S}_{\ell^j/2} & 0 \\ (1+\varepsilon)^{-1/2}(B_j)_{y^j} & (1+\varepsilon)^{1/2}\mathcal{S}_{-\ell^j/2} \end{bmatrix} : \varepsilon > 0, \ell^j \in \mathbb{R}, y^j \in \mathbb{R}^2 \tag{19}$$

$$(B_j)_{y^j} = \begin{bmatrix} 0 & (y^j)_1 \\ (y^j)_1 & (y^j)_2 \end{bmatrix}, \quad y^j = ((y^j)_1, (y^j)_2) \in \mathbb{R}^2,\ \mathcal{S}_{\ell^j} = \begin{bmatrix} 1 & \ell^j \\ 0 & 1 \end{bmatrix}, \quad \ell^j \in \mathbb{R} \tag{20}$$

The matrices $\mathcal{S}_{\ell^j}$ is called shearing matrix. We use the letters TDS because the restriction of the metaplectic representation to it gives rise to translation, dilation, and shearing operators. This fact will be discussed. The main idea of the proof is to reduce the two-dimensional condition (16) to the one-dimensional analogue that arises from a reproducing subgroup of $\mathbb{R}^2 \rtimes$ SL(2, ℝ) and to a reproducing condition for a window function of another reproducing subgroup of$\mathbb{R}^2 \rtimes$ SL(2, ℝ). It was proven in [8, Thm.2.1] that, up to conjugation, there are exactly five reproducing subgroups of $\mathbb{R}^2 \rtimes$Sp(1+ε, ℝ). We are interested in the following two:

$$H_0 = \left\{\left(\begin{bmatrix} 1+\varepsilon \\ 1+\varepsilon \end{bmatrix}, I\right), (1+\varepsilon), (1+\varepsilon) \in \mathbb{R}\right\}$$

$$H_1 = \left\{\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ a+\varepsilon & 1 \end{bmatrix}\begin{bmatrix} a^{-1/2} & 0 \\ 0 & a^{1/2} \end{bmatrix}\right), \varepsilon > -a, (a+\varepsilon) \in \mathbb{R}\right\}$$

Sequence of functions $(\phi_j)_0$ are reproducing for H_0 if

$$(\phi_j)_0 \in L^2(\mathbb{R}), \text{ and } \|(\phi_j)_0\|_2 = 1 \tag{21}$$

While the sequence of functions $(\phi_j)_1$ are reproducing for H_1if and only if $(\phi_j)_1 \in L^2(\mathbb{R})$ and

$\int_0^\infty \sum_j \left|(\phi_j)_1(\sum_r x_r^j)\right|^2 \frac{d(\sum_r x_r^j)}{(\sum_r x_r^j)^2} = \int_0^\infty \sum_j \left|(\phi_j)_1(-\sum_r x_r^j)\right|^2 \frac{d(\sum_r x_r^j)}{(\sum_r x_r^j)^2} = \frac{1}{2},$

$\int_0^\infty (\phi_j)_1(\sum_r x_r^j)\overline{(\phi_j)_1(-\sum_r x_r^j)} \frac{d(\sum_r x_r^j)}{(\sum_r x_r^j)^2} = 0$ (22)

Clearly,$H_0 \simeq \mathbb{R}^{2(1+\varepsilon)}$ so that its Haar measure is the Lebesgue measure $d(1+\varepsilon)d(1+\varepsilon)$. The group H_1 is the only reproducing subgroup that lies entirely inside Sp(1, ℝ) = SL(2, ℝ),and it is isomorphic to the "$a(\sum_r x_r^j) + (a+\varepsilon)$" group. Its right Haar measure is $dad(a+\varepsilon)/a$

Observe that (16) can be rewritten in terms of the right Haar measure $d_r h_j$ as

$$\int_H \sum_j W_{\phi_j}^j \left(h_j^{-1} \cdot (\sum_r x_r^j, \sum_r \xi_r^j)\right) dh_j = \int_H \sum_j W_{\phi_j}^j \left(h_j \cdot (\sum_r x_r^j, \sum_r \xi_r^j)\right) d_r h_j = 1$$

leading to the following alternative formulation that H_0 and $H_1$is admissible$\int_{\mathbb{R}^2} \sum_j W_{(\phi_j)_0}^j \left(\sum_r x_r^j + (1+\varepsilon), \sum_r \xi_r^j + (1+\varepsilon)\right) d(1+\varepsilon)d(1+\varepsilon) = 1,$

for a. e. $(\sum_r x_r^j, \sum_r \xi_r^j) \in \mathbb{R}^2$ (23)

$\int_{\mathbb{R}} \int_0^\infty \sum_j W_{\phi_j}^j \left(a^{-1/2}(\sum_r x_r^j), a^{-1/2}(a+\varepsilon)\sum_r x_r^j + a^{1/2}(\sum_r \xi_r^j)\right) \frac{da}{a} d(a+\varepsilon) = 1$

for a.e. $(\sum_r x_r^j, \sum_r \xi_r^j) \in \mathbb{R}^2$ (24)

We can finally show that TDS(2) is reproducing.

Theorem 9. Let $(\phi_j)_0, (\phi_j)_1 \in L^2(\mathbb{R})$ be reproducing sequence of functions for the subgroups H_0 and H_1, respectively. Then, the window sequence of functions ϕ_j defined by

$\phi_j(\sum_r x_r^j, \sum_r \xi_r^j) = \frac{1}{2}((\phi_j)_0 \otimes (\widetilde{\phi}_j)_1(\sum_r x_r^j, \sum_r \xi_r^j), (\sum_r x_r^j, \sum_r \xi_r^j) \in \mathbb{R}^2$ (25)

where $(\widetilde{\phi}_j)_1(y^j) = y^j(\phi_j)_1(y^j)$, are reproducing sequence of function for TDS(2), i.e., TDS(2) are reproducing subgroups.

Proof. Sinceit is convenient to deal with absolutely convergent integrals, let us first assume

$(\phi_j)_0, (\phi_j)_1 \in \mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$Notice that the assumptions (14) and (15) are trivially satisfied. Hence, it remains to verify the admissibility condition (16), i.e.

$\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_0^\infty \sum_j W_{\phi_j}^j \left((z^j,\zeta^j) \quad \left({}^{(1+\varepsilon)}(A_j)_{(1+\varepsilon),\ell^j,y^j}\right)\right) \frac{d(1+\varepsilon)}{(1+\varepsilon)} d\ell^j d(y^j)_1 d(y^j)_2 = 1,$

$a.e. (z^j, \zeta^j) \in \mathbb{R}^4$ (26)

where $\left(\frac{1}{1+\varepsilon}\right) d(1+\varepsilon)\, d\ell^j d(y^j)_1 d(y^j)_2$ are the right Haar measures of TDS(2). First, we compute

$(z^j,\zeta^j)\left({}^{(1+\varepsilon)}(A_j)_{(1+\varepsilon),\ell^j,y^j}\right)$ with $z^j = ((z^j)_1, (z^j)_2), \zeta^j = ((\zeta^j)_1, (\zeta^j)_2) \in \mathbb{R}^4$ that are

$(z^j,\zeta^j)({}^{(1+\varepsilon)}(A_j)_{(1+\varepsilon),\ell^j,y^j}) =$
$\Big((1+\varepsilon)^{-1/2}\left((z^j)_1 + \frac{\ell^j}{2}(z^j)_2\right), (1+\varepsilon)^{-1/2}(z^j)_2, (1+\varepsilon)^{-1/2}(y^j)_1(z^j)_2 + (1+\varepsilon)^{1/2}(\zeta^j)_1, (1+\varepsilon)^{-1/2}\left[(y^j)_1\left((z^j)_1 + \frac{\ell^j}{2}\right)(z^j)_2 + (y^j)_2(z^j)_2\right] + (1+\varepsilon)^{1/2}\left(-\frac{\ell^j}{2}(\zeta^j)_1 + (\zeta^j)_2\right)\Big)$

Secondly, we use Lemma 5 to compute the Wigner distributions $W_{\phi_j}^j$ of the sequence of functions ϕ_j, defined in (25), at the points $(z^j,\zeta^j)\,({}^{(1+\varepsilon)}(A_j)_{(1+\varepsilon),\ell^j,y^j})$:

$W_{\phi_j}^j(z^j,\zeta^j)^{(1+\varepsilon)}(A_j)_{(1+\varepsilon),\ell^j,y^j} = \frac{1}{2}W_{(\phi_j)_0}^j\left((1+\varepsilon)^{-1/2}\left((z^j)_1 + \frac{\ell^j}{2}(z^j)_2\right), (1+\varepsilon)^{-1/2}(y^j)_2(z^j)_2 + (1+\varepsilon)^{1/2}(\zeta^j)_1\right)$

$\times W^{j}_{(\widetilde{\phi_j})_1}\left((1+\varepsilon)^{-1/2}(z^j)_2,(1+\varepsilon)^{-1/2}\left[(y^j)_1\left((z^j)_1+\frac{\ell^j}{2}(z^j)_2\right)+(y^j)_2(z^j)_2\right]+(1+\varepsilon)^{1/2}\left(-\frac{\ell^j}{2}(\zeta^j)_1+(\zeta^j)_2\right)\right)$

Lemma 6 for the Wigner distributions $W^{j}_{(\widetilde{\phi_j})_1}$ gives

$$W^{j}_{(\widetilde{\phi_j})_0}\left((1+\varepsilon)^{-1/2}(z^j)_2,(1+\varepsilon)^{-1/2}\left[(y^j)_1\left((z^j)_1+\frac{\ell^j}{2}(z^j)_2\right)+(y^j)_2(z^j)_2\right]+(1+\varepsilon)^{1/2}\left(-\frac{\ell^j}{2}(\zeta^j)_1+(\zeta^j)_2\right)\right)$$
$$=\frac{(z^j)_2^2}{(1+\varepsilon)}W^{j}_{(\phi_j)_1}\left((1+\varepsilon)^{-1/2}(z^j)_2,(1+\varepsilon)^{-1/2}\left[(y^j)_1\left((z^j)_1+\frac{\ell^j}{2}(z^j)_2\right)+(y^j)_2(z^j)_2\right]+(1+\varepsilon)^{1/2}\left(-\frac{\ell^j}{2}(\zeta^j)+(\zeta^j)_2\right)\right)$$
$$+\frac{1}{16\pi^2(1+\varepsilon)}\partial^2_{\zeta^j}W^{j}_{(\phi_j)_1}\left((1+\varepsilon)^{-1/2}(z^j)_2,(1+\varepsilon)^{-1/2}\left[(y^j)_1\left((z^j)_1+\frac{\ell^j}{2}(z^j)_2\right)+(y^j)_2(z^j)_2\right]+(1+\varepsilon)^{1/2}\left(-\frac{\ell^j}{2}(\zeta^j)_1+(\zeta^j)_2\right)\right) \quad (27)$$

The last term comes from the chain rule, for if

$$\sum_r \xi^j_r=\left(\sum_r \xi^j_r\right)((z^j)_1,(z^j)_2,(\zeta^j)_1(\zeta^j)_2)=(1+\varepsilon)^{-1/2}\left[(y^j)_1\left((z^j)_1+\frac{\ell^j}{2}(z^j)_2\right)+(y^j)_2(z^j)_2\right]+(1+\varepsilon)^{1/2}\left(-\frac{\ell^j}{2}(\zeta^j)_1+(\zeta^j)_2\right)$$

Then

$$\partial^2_{(\sum_r \xi^j_r)}W^j_{g^j}\left((1+\varepsilon)^{-1/2}(z^j)_2,\sum_r \xi^j_r\right)=$$
$$\left(\frac{1}{1+\varepsilon}\right)(\partial)^2_{\zeta^j}W^j_{g^j}\left((1+\varepsilon)^{-1/2}(z^j)_2,(1+\varepsilon)^{-1/2}\left[(y^j)_1\left((z^j)_1+\frac{\ell^j}{2}(z^j)_2\right)+(y^j)_2(z^j)_2\right]+(1+\varepsilon)^{1/2}\left(-\frac{\ell^j}{2}(\zeta^j)_1+(\zeta^j)\right)\right)$$

Next, we compute the integral on the left hand side of (26). From (27), this amounts to computing the sum of two integrals. We deal with each of them separately. Since (16) holds true for a.e. $((z^j)_1,(z^j)_2,(\zeta^j)_1,(\zeta^j)_2)\in\mathbb{R}^4$, we can assume $(z^j)_2\neq 0$. Computation of the integral. By performing the change of variables

$$\acute{\ell}^j=(1+\varepsilon)^{-1/2}(z^j)_2\,\ell^j/2,\ (\acute{y}^j)_1=(1+\varepsilon)^{-1/2}(z^j)_2(y^j)_1,\ d\ell^j d(y^j)_1=\frac{2(1+\varepsilon)}{(z^j)_2^2}\,d\acute{\ell}^j d(\acute{y}^j)_1$$

We obtain

$$I_1=\frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_0^\infty\sum\ W^j_{(\phi_j)_0}\left((1+\varepsilon)^{-1/2}\left((z^j)_1+\frac{\ell^j}{2}(z^j)_2\right),(1+\varepsilon)^{-1/2}(y^j)_1(z^j)_2+(1+\varepsilon)^{1/2}(\zeta^j)_1\right)$$
$$\times$$
$$\frac{(z^j)_2^2}{(1+\varepsilon)}W^j_{(\phi_j)_1}\left((1+\varepsilon)^{-1/2}(z^j)_2,(1+\varepsilon)^{-1/2}\left[(y^j)_1\left((z^j)_1+\frac{\ell^j}{2}(z^j)_2\right)+(y^j)_2(z^j)_2\right]+\right.$$
$$\left.(1+\varepsilon)^{1/2}\left(-\frac{\ell^j}{2}(\zeta^j)_1+(\zeta^j)_2\right)\right)\frac{d(1+\varepsilon)}{(1+\varepsilon)}d\ell^j d(y^j)_1 d(y^j)_2$$

$$=\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_0^\infty\sum_j W^j_{(\phi_j)_0}\left((1+\varepsilon)^{-1/2}(z^j)_1+\acute{\ell}^j,(\acute{y}^j)_1+(1+\varepsilon)^{1/2}(\zeta^j)_1\right)$$
$$\times$$
$$W^j_{(\phi_j)_1}\left((1+\varepsilon)^{-1/2}(z^j)_2,(1+\varepsilon)^{-1/2}(z^j)_2\left[\frac{(1+\varepsilon)^{1/2}(\acute{y}^j)_1(z^j)_1}{(z^j)_2^2}+\frac{(1+\varepsilon)(\acute{y}^j)_1\acute{\ell}^j}{(z^j)_2^2}+(z^j)_2-\frac{(1+\varepsilon)^{3/2}\acute{\ell}^j(\zeta^j)_1}{(z^j)_2^2}\right]+\right.$$
$$\left.(1+\varepsilon)^{1/2}(\zeta^j)_2\right)\frac{d(1+\varepsilon)}{(1+\varepsilon)}d\acute{\ell}^j\,d(\acute{y}^j)_1\,d(y^j)_2$$

Integrating first with respect to the variable $(y^j)_2$, and making the change of variables

$$(\acute{y}^j)_1=\frac{(1+\varepsilon)^{1/2}(\acute{y}^j)_1(z^j)_1}{(z^j)_2^2}+\frac{(1+\varepsilon)(\acute{y}^j)_1\acute{\ell}^j}{(z^j)_2^2}+(y^j)_2-\frac{(1+\varepsilon)^{3/2}\acute{\ell}^j(\zeta^j)_1}{(z^j)_2^2},d(y^j)_2=d(\acute{y}^j)_1$$

The admissibility condition (23) for the subgroup H_0, with $(1+\varepsilon)=\acute{\ell}^j$, $(1+\varepsilon)=(\acute{y}^j)_1$ and $\left(z^j,\sum_r \xi^j_r\right)=((1+\varepsilon)^{-1/2}(z^j)_1,(1+\varepsilon)^{1/2}(\zeta^j)_1)$ shows that I_1 is equal to

$\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_0^{\infty}\sum_j W^j_{(\phi_j)_0}\left((1+\varepsilon)^{-1/2}(z^j)_1+\acute{\ell}^j,(1+\varepsilon)^{1/2}(\zeta^j)_1+(\acute{y}^j)_2\right)W^j_{(\phi_j)_1}\left((1+\varepsilon)^{-1/2}(z^j)_2,(1+\varepsilon)^{-1/2}(z^j)_2(\acute{y}^j)_1+(1+\varepsilon)^{1/2}(\zeta^j)_2\right)\frac{d(1+\varepsilon)}{1+\varepsilon}d\acute{\ell}^j d(\acute{y}^j)_1\, d(y^j)_2$

$=\int_{\mathbb{R}}\int_0^{\infty}\sum_j W^j_{(\phi_j)_1}\left((1+\varepsilon)^{-1/2}(z^j)_2,(1+\varepsilon)^{-1/2}(z^j)_1(\acute{y}^j)_2+(1+\varepsilon)^{1/2}(\zeta^j)_2\right)\frac{d(1+\varepsilon)}{1+\varepsilon}d(\acute{y}^j)_2$

Finally, the admissibility condition (24) for the subgroup H_0, with a = $1+\varepsilon$, a + ε =$(\acute{y}^j)_2$

Gives

$$\int_{\mathbb{R}}\int_0^{\infty}\sum_j W^j_{(\phi_j)_1}\left((1+\varepsilon)^{-1/2}(z^j)_2,(1+\varepsilon)^{-1/2}(z^j)_2(\acute{y}^j)_2+(1+\varepsilon)^{1/2}(\zeta^j)_2\right)\frac{d(1+\varepsilon)}{(1+\varepsilon)}d(\acute{y}^j)_2=1,$$
$$for\ a.e.\ ((z^j)_2,(\zeta^j)_2)\in\mathbb{R}^2 \quad (28)$$

Computation of the second integral. All we are left to show is that

$I_2=\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_0^{\infty}\sum_j W^j_{(\phi_j)_0}\left((1+\varepsilon)^{-1/2}\left((z^j)_1+\frac{\ell^j}{2}(z^j)_2\right),(1+\varepsilon)^{-1/2}(y^j)_1(z^j)_2+(1+\varepsilon)^{1/2}(\zeta^j)_1\right)\frac{1}{(1+\varepsilon)}$

$\times\,\partial^2_{(\sum_r \xi^j_r)_2}W^j_{(\phi_j)_1}\left((1+\varepsilon)^{-1/2}(z^j)_2,(1+\varepsilon)^{-1/2}\left[(y^j)_1\left((z^j)_1+\frac{\ell^j}{2}(z^j)_2\right)+(y^j)_2(z^j)_2\right]+(1+\varepsilon)^{1/2}\left(-\frac{\ell^j}{2}(\zeta^j)_1+(\zeta^j)_2\right)\right)\frac{d(1+\varepsilon)}{1+\varepsilon}d\ell^j d(y^j)_1 d(y^j)_2$

Vanishes, By performing the same computations as for I_1, we obtain

$I_2=\sum_j\frac{1}{(z^j)_2^2}\partial^2_{(\sum_r\xi^j_r)_2}\left(\int_{\mathbb{R}}\int_0^{\infty}W^j_{(\phi_j)_1}\left((1+\varepsilon)^{-1/2}(z^j)_2,(1+\varepsilon)^{-1/2}(z^j)_2(\acute{y}^j)_2+(1+\varepsilon)^{1/2}(\zeta^j)_2\right)\right)$

$\cdot\left(\frac{d(1+\varepsilon)}{(1+\varepsilon)}d(\acute{y}^j)_2\right)$

Admissibility implies that the expression inside the barackets is equal to 1 for a.e.

$(\zeta^j)_2\in\mathbb{R}^{(1+\varepsilon)}$, so that $I_2=0$. By a density argument, the sequence of windows functions $(\phi_j)_0,(\phi_j)_1\in\mathcal{S}(\mathbb{R})$ can be replaced by rougher reproducing function in $L^2(\mathbb{R})$, thereby finishing the proof.

Condition with Wavelet Theory

We come closer to the group theory that lies behind the construction of two-dimensional wavelets. The analysis of oriented features in images requires more flexible objects than the wavelets arising from the tensor product of the usual one-dimensional wavelets. Answers to this problem, in the context of signal processing, have been provided by frame systems of directional functions with excellent angular selectivity, the frames of curvelets and contourlets (Candes *et al.,* 2001; Do and Vetterli). Both make use of translation and dilation operations, and while the curvelet approach obtains directional selectivity by a construction that requires a rotation operation, the contourlet setup uses a sheering operation. Although the results do not have direct implications in either curve let or contourlet analysis, we point out that they appear to be connected from the point of view of group theory, that is, by looking at the restriction of the metaplectic representation to two admissible subgroups of $Sp(2,\mathbb{R})$, namely $SIM(2)$and (2).

The main results in this section are Theorem 12 and Theorem 14, (see. e.g.,(Cordero *et al.,* 2000)).

The group $SIM(2)$ and its natural representation

We prove Theorem 12. The similitude group SIM(2) of the plane $\mathbb{R}^2$is the group generated by translations, rotations, and dilations. (a survey on the topic and the related two-dimensional directional wavelets is in [1]). More precisely, for a real angle θ_j put

$$(R)_{\theta_j}=\begin{bmatrix}\cos\theta_j & \sin\theta_j\\ -\sin\theta_j & \cos\theta_j\end{bmatrix} \quad (29)$$

The standard 2 × 2 rotation matrix. Then SIM(2) consists of all the 3 × 3 matrices

$$T\big(1+\varepsilon, y^j, \theta_j\big) = \begin{bmatrix} (1+\varepsilon)(R)_{-\theta_j} & y^j \\ 0 & 1 \end{bmatrix}$$

Where $\varepsilon > -1$, y^j is a column vector in $\mathbb{R}^2$ and $\theta_j \in [0, 2\pi)$. The product in SIM(2) is just matrix product and a simple calculation yields

$$T\big(1+\varepsilon, y^j, \theta_j\big)T\big(s^j, \theta_j, \phi_j\big) = T\left((1+\varepsilon)s^j, y^j + (1+\varepsilon)(R)_{\theta_j}(z^j), \theta_j + \phi_j\right) \tag{30}$$

Formally, the action of SIM(2) on $\mathbb{R}^2$ is obtained by viewing $\mathbb{R}^2$ as one of the affine charts in $\mathbb{RP}^2$, namely

$$\mathbb{R}^2 \simeq \left\{\begin{bmatrix} \sum_r x_r^j \\ 1 \end{bmatrix} : \sum_r x_r^j \in \mathbb{R}^2\right\} \subset \mathbb{RP}^2$$

In other words, SIM(2) acts on $\mathbb{RP}^2$ preserving this affine chart:

$$T\big(1+\varepsilon, y^j, \theta_j\big)\begin{bmatrix} \sum_r x_r^j \\ 1 \end{bmatrix} = \begin{bmatrix} (1+\varepsilon)(R)_{-\theta_j} & y^j \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sum_r x_r^j \\ 1 \end{bmatrix} = \begin{bmatrix} (1+\varepsilon)(R)_{-\theta_j}(\sum_r x_r^j) + y^j \\ 1 \end{bmatrix}$$

The wavelet representation ν of SIM(2) on $L^2(\mathbb{R}^2)$ is defined as follows:

$$\nu(1+\varepsilon, y^j, \theta_j) f_j\left(\textstyle\sum_r x_r^j\right) = \left(\tfrac{1}{1+\varepsilon}\right) f_j\left(\left(\tfrac{1}{1+\varepsilon}\right)\left((R)_{\theta_j}\left(\textstyle\sum_r x_r^j - y^j\right)\right)\right),$$

where $\nu(1+\varepsilon, y^j, \theta_j)$ stands for $\nu(T(1+\varepsilon, y^j, \theta_j))$. Notice that if we transpose rotations, dilations and translations to functions by

$$\big((R_j)_{\theta_j} f_j\big)\left(\textstyle\sum_r x_r^j\right) = f_j\left((R)_{\theta_j}\textstyle\sum_r x_r^j\right), \big(D_{(1+\varepsilon)} f_j\big)\left(\textstyle\sum_r x_r^j\right) = \left(\tfrac{1}{1+\varepsilon}\right) f_j\left(\left(\tfrac{1}{1+\varepsilon}\right)\textstyle\sum_r x_r^j\right), \big(T_{y^j} f_j\big)\left(\textstyle\sum_r x_r^j\right) = f_j\left(\textstyle\sum_r x_r^j - y^j\right)$$

then $\nu(1+\varepsilon, y^j, \theta_j) f_j = (T_{y^j}(R_j)_{\theta_j} D_{(1+\varepsilon)}) f_j$. The representation ν is known to be irreducible on $L^2(\mathbb{R}^2)$ and it gives rise to a reproducing formula. The sequence of wavelets ϕ_j are reproducing if

$$\int_{\mathbb{R}^2} \sum_j \frac{\left|\hat{\phi}_j(\sum_r \xi_r^j)\right|^2}{\left|\sum_r \xi_r^j\right|^2} d\left(\textstyle\sum_r \xi_r^j\right) < +\infty$$

For our purposes however, it is convenient to view ν in the frequency domain, that is, to compose it with the Fourier transform $\mathcal{F}$. We shall therefore write

$$\pi\big(1+\varepsilon, y^j, \theta_j\big) f_j(u_j) = \big(\mathcal{F} \circ \nu(1+\varepsilon, y^j, \theta_j) f_j\big)(u_j) = (1+\varepsilon) e^{-2\pi i \langle y^j, u_j\rangle} f_j\left((1+\varepsilon)(R)_{\theta_j} u_j\right) \tag{31}$$

The group SIM(2) as a subgroup of Sp(2, $\mathbb{R}$) and the action on the lagrange manifold. We adopt the following notation. If $y^j = ((y^j)_1, (y^j)_2) \in \mathbb{R}^2$, we put

We adopt the following notation. If $y^j = ((y^j)_1, (y^j)_2) \in \mathbb{R}^2$, we put

$$\Sigma_{y^j} = \begin{bmatrix} (y^j)_1 & (y^j)_2 \\ (y^j)_2 & -(y^j)_1 \end{bmatrix} \tag{32}$$

a 2×2 symmetric and traceless matrix. Consider the two subgroups of Sp(2, $\mathbb{R}$):

$$G_0 = \left\{(g^j)_0(1+\varepsilon, y^j) = \begin{bmatrix} (1+\varepsilon)^{-1/2} & 0 \\ (1+\varepsilon)^{-1/2}\Sigma_{y^j} & (1+\varepsilon)^{1/2} \end{bmatrix} : \varepsilon > -1, y^j \in \mathbb{R}^2\right\} \tag{33}$$

$$K = \left\{k(\theta_j) = \begin{bmatrix} (R)_{-\theta_j/2} & 0 \\ 0 & (R)_{-\theta_j/2} \end{bmatrix} : \theta_j \in [0, 4\pi]\right\} \tag{34}$$

It is straightforward to check that:

$$(g^j)_0(1+\varepsilon, y^j)(g^j)_0(s^j, z^j) = (g^j)_0\left((1+\varepsilon)s^j, y^j + (1+\varepsilon)z^j\right)$$

$$k(\theta_j)(g^j)_0(s^j, z^j)k(\theta_j)^{-1} = (g^j)_0(s^j, (R)_{-\theta}z^j) \tag{35}$$

the latter being immediate from

$$(R)_{-\theta_j/2}\,\Sigma_{(\Sigma_r x_r^j)}(R)_{\theta_j/2} = \Sigma_{(R)_{-\theta_j}(\Sigma_r x_r^j)} \tag{36}$$

The equality (35) shows that K normalizes G_0 and hence that G_0 K inherits the structure of a semidirect products, where the product law are given by

$(g^j)_0\ (1+\varepsilon,\ y^j)k(\theta_j)\cdot(g^j)_0\ (s^j,\ z^j)k(\phi_j) = (g^j)_0\ (1+\varepsilon,\ y^j)\ [k(\theta_j)(g^j)_0\ (s^j,\ z^j)k(\theta_j\ )^{-1}\ ]k(\theta_j)k(\phi_j)$

$= (g^j)_0\ (1+\varepsilon,\ y^j)(g^j)_0\ (s^j,\ (R)_{-\theta_j}z^j)\ k(\theta_j+\phi_j)$

$= (g^j)_0\ (\ (1+\varepsilon)\ s^j,\ y^j + (1+\varepsilon)(R)_{-\theta_j}z^j)\ k(\theta_j+\phi_j).$

Of course, $G_0 \rtimes$ K is a subgroup of Sp(2, ℝ). Further, G_0 is in $G_0\rtimes$ Kobviously $G_0\rtimes K/G_0 \simeq$ K.
We shall write

$$g^j(1+\varepsilon, y^j, \theta_j) = (g^j)_0(1+\varepsilon, y^j)k(\theta_j) = \begin{bmatrix}(1+\varepsilon)^{-1/2} & 0\\ (1+\varepsilon)^{-1/2}\Sigma_{y^j} & (1+\varepsilon)^{1/2}\end{bmatrix}\begin{bmatrix}(R)_{-\theta_j/2} & 0\\ 0 & (R)_{-\theta_j/2}\end{bmatrix} = \begin{bmatrix}(1+\varepsilon)^{-1/2}(R)_{-\theta_j/2} & 0\\ (1+\varepsilon)^{-1/2}\Sigma_{y^j}(R)_{-\theta_j/2} & (1+\varepsilon)^{1/2}(R)_{-\theta_j/2}\end{bmatrix}$$

Therefore

$g^j\ (1+\varepsilon,\ y^j,\ \theta_j)g^j\ (s^j,\ z^j, \phi_j) = g^j\ ((1+\varepsilon)\ s^j,\ y^j + (1+\varepsilon)\ (R)_{-\theta_j}z^j,\ (\theta_j+\phi_j)$

The mappings

$g^j\ (1+\varepsilon,\ y^j,\ \theta_j) \longmapsto T\ (1+\varepsilon,\ y^j,\ \theta_j \bmod 2\pi\ )$

which exhibits $G_0 \rtimes K$ as canonically isomorphic to SIM(2) (see (30))

Next, we identify the action of SIM(2) on $\mathbb{R}^2$ with the action of $G_0\rtimes$ K on a suitable two-dimensional cell C of the Lagrange manifold L($\mathbb{R}^4$). The Lagrange manifold is defined as the set of maximal isotropic planes in$\mathbb{R}^4$, namely the two-dimensional linear subspaces of $\mathbb{R}^4$ that enjoy the following properties:

If$\sum_r x_r^j,\ y^j \in \mathcal{L}$, then $\omega(\sum_r x_r^j,\ y^j) = 0$. This set inherits the manifold structure of a three-dimensional homogeneous spaces of Sp(2, ℝ). Indeed, letus represent planes in $\mathbb{R}^4$as 4 × 2 matrices via

$$\mathcal{L}(A_j, B_j) = span\begin{bmatrix}A_j\\ B_j\end{bmatrix}, \qquad A_j, B_j \in M_2(\mathbb{R}),\ rank\begin{bmatrix}A_j\\ B_j\end{bmatrix} = 2 \tag{37}$$

Under the identification (37), two 4 × 2 full-rank matrices identify the same plane if and only if they differ by right multiplication by some $g^j \in$ GL(2, ℝ). Such a 4 × 2 full-rank matrix represents a Lagrangian plane if and only if $\left({}^{(1+\varepsilon)}A_j\right)B_j$ is symmetric. Also, its columns form an orthonormal set if and only if

$\left({}^{(1+\varepsilon)}A_j\right)A_j + \left({}^{(1+\varepsilon)}B_j\right)B_j = I$. In this case,

$$g^j(A_j, B_j) = \begin{bmatrix}A_j & -B_j\\ B_j & A_j\end{bmatrix} \in Sp(2,\mathbb{R})$$

and $g^j(A_j,\ B_j)$ carries the "base" Lagrangian plane $\mathcal{L}_0 = \mathcal{L}(I, 0)$ onto $\mathcal{L}(A_j,\ B_j)$. In general, Sp(2, ℝ) acts on Lagrangian spaces from the left, by matrix multiplication on the spanning column vectors. Since we know that $g^j(A_j,\ B_j)\cdot\mathcal{L}_0 = \mathcal{L}\ (A_j,\ B_j)$, the Sp(2, ℝ)-action is transitive on $L(\mathbb{R}^4)$ and the stabilizer at $\mathcal{L}_0$ is the subgroup

$$U = \left\{ \begin{bmatrix} A_j & \Sigma^j A_j \\ 0 & {}^{(1+\varepsilon)}A_j^{-1} \end{bmatrix} : A_j \in GL(2,\mathbb{R}), \Sigma^j \; symmetric \right\} \subset Sp(2,\mathbb{R})$$

Thus, $L(\mathbb{R}^4) \simeq Sp(2,\mathbb{R})/U$. An open set in $L(\mathbb{R}^4)$ that contains the base point $\mathcal{L}_0$ is $L_0 = \{ \mathcal{L}(\Sigma^j) := \mathcal{L}(I, \Sigma^j) : \Sigma^j \text{symmetric} \}$, and is diffeomorphic to $\mathbb{R}^3$under the identification $\Sigma^j \leftrightarrow \mathcal{L}(\Sigma^j)$. We put
$\mathcal{C} = \left\{ \mathcal{L}(\sum_r x_r^j) = \mathcal{L}\left(\Sigma^j_{x^j}\right) : \sum_r x_r^j \in \mathbb{R}^2 \right\}$

the two-dimensional slice inside L_0 identified with the traceless symmetric matrices.

Proposition 10. The action of SIM(2) on $\mathbb{R}^2$corresponds to the natural action of $G_0 \rtimes K$ on C inside the Lagrange manifold $L(\mathbb{R}^4)$.
Proof. Allowing right multiplication by $(1+\varepsilon)^{1/2}(R)_{\theta_j/2}$(and using (36), we compute

$$g^j(1+\varepsilon, y^j, \theta_j) \cdot \mathcal{L}(\textstyle\sum_r x_r^j)$$

$$= span\left(\begin{bmatrix} (1+\varepsilon)^{-1/2}(R)_{-\theta_j/2} & 0 \\ (1+\varepsilon)^{-1/2}\Sigma^j_{y^j}(R)_{-\theta/2} & (1+\varepsilon)^{1/2}(R)_{-\theta_j/2} \end{bmatrix} \begin{bmatrix} 1 \\ \Sigma^j_{(\sum_r x_r^j)} \end{bmatrix} \left[(1+\varepsilon)^{1/2}(R)_{\theta_j/2}\right] \right)$$

$$= span\begin{bmatrix} 1 \\ \Sigma^j_{y^j} + (1+\varepsilon)(R)_{-\theta/2}\Sigma^j_{(\sum_r x_r^j)}(R)_{\theta_j/2} \end{bmatrix}$$

$$= span\begin{bmatrix} 1 \\ \Sigma^j_{y^j} + (1+\varepsilon)\Sigma^j_{((R)_{\theta_j})(\sum_r x_r^j)} \end{bmatrix}$$

$$= span\begin{bmatrix} 1 \\ \Sigma^j_{y^j+(1+\varepsilon)(R)_{-\theta_j}(\sum_r x_r^j)} \end{bmatrix}$$

$$= \mathcal{L}(y^j + (1+\varepsilon)(R)_{-\theta_j}(\textstyle\sum_r x_r^j))$$

Therefore, under the canonical homomorphism of $G_0 \rtimes K$ onto SIM(2), the action of SIM(2) on $\mathbb{R}^2$corresponds to the natural action of $G_0 \rtimes K$ on $\mathcal{C}$.

From compute the metaplectic representation μ on $G_0 \rtimes K$. We start from a simple observation. Every g^j $(1+\varepsilon, \theta_j, y^j) \in G_0 \rtimes K$ decomposes as the product of a block-diagonal matrix and a block-lower triangular matrix, both in Sp(2, ℝ), as follows:

$$\begin{bmatrix} (1+\varepsilon)^{-1/2}(R)_{-\theta_j/2} & 0 \\ (1+\varepsilon)^{-1/2}\Sigma^j_{y^j}(R)_{-\theta_j/2} & (1+\varepsilon)^{1/2}(R)_{-\theta_j/2} \end{bmatrix} = \begin{bmatrix} (1+\varepsilon)^{-1/2}(R)_{-\theta_j/2} & 0 \\ 0 & (1+\varepsilon)^{1/2}(R)_{-\theta_j/2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (1+\varepsilon)^{-1/2}\Sigma^j_{y^j}(R)_{-\theta_j/2} & 1 \end{bmatrix}$$

We rewrite this as g^j $(1+\varepsilon, y^j, \theta_j) = D(1+\varepsilon, \theta_j)\, L(1+\varepsilon, y^j, \theta_j)$. Owing to (6) and (7), we have

$$\mu\left(g^j(1+\varepsilon, y^j, \theta_j)\right) f_j(\textstyle\sum_r x_r^j) = \mu\left(D(1+\varepsilon, \theta_j) L(1+\varepsilon, y^j, \theta_j)\right) f_j(\sum_r x_r^j)$$

$$= det((1+\varepsilon)^{-1/2}(R)_{-\theta_j/2})^{-1/2}\, \mu\left(L(1+\varepsilon, y^j, \theta_j)\right) f_j((1+\varepsilon)^{1/2}(R)_{\theta_j/2}(\textstyle\sum_r x_r^j))$$

$$= (1+\varepsilon)^{1/2} exp\left(-i\pi \left\langle \left[(1+\varepsilon)^{-1/2}(R)_{\theta_j/2}\Sigma^j_{y^j}(R)_{-\theta_j/2}\right](1+\varepsilon)^{1/2}(R)_{\theta_j/2}(\textstyle\sum_r x_r^j), (1+\varepsilon)^{1/2}(R)_{\theta_j/2}(\sum_r x_r^j)\right\rangle\right) f_j((1+\varepsilon)^{1/2}(R)_{\theta_j/2}(\sum_r x_r^j))$$

$$= (1+\varepsilon)^{1/2} exp\left(-i\pi \left\langle (1+\varepsilon)^{-1/2}(R)_{\theta_j/2}\Sigma^j_{y^j}(\textstyle\sum_r x_r^j), (1+\varepsilon)^{1/2}(R)_{\theta_j/2}(\sum_r x_r^j)\right\rangle\right) \times\, f_j((1+\varepsilon)^{1/2}(R)_{\theta_j/2}(\sum_r x_r^j))$$

$$= (1+\varepsilon)^{1/2} e^{-i\pi\langle \Sigma^j_{y^j}(\sum_r x_r^j), \sum_r x_r^j\rangle} f_j((1+\varepsilon)^{1/2}(R)_{\theta_j/2}(\textstyle\sum_r x_r^j))$$

That are

$$\mu\left(g^j(1+\varepsilon, y^j, \theta_j)\right) f_j(\textstyle\sum_r x_r^j) = (1+\varepsilon)^{1/2} e^{-i\pi\langle \Sigma^j_{y^j}(\sum_r x_r^j), \sum_r x_r^j\rangle} f_j\left((1+\varepsilon)^{1/2}(R)_{\theta_j/2}(\sum_r x_r^j)\right) \quad (38)$$

The intertwining operator and the equivalence

Consider the mapping

$$\Phi : \dot{\mathbb{R}} \times \mathbb{R}_+ \to \mathbb{R}^2\backslash\{(0,0)\}, \sum_r x_r^j \mapsto \left(\frac{(\sum_r x_r^j)_1^2-(\sum_r x_r^j)_2^2}{2}, (\sum_r x_r^j)_1(\sum_r x_r^j)_2\right) \quad (39)$$

Its properties are described in the next propositions.

Proposition 11. Let f_j be an even function defined on $\mathbb{R}^2$. The mapping (39) is a diffeomorphism that satisfies:

(a) The Jacobian of Φat $\sum_r x_r^j \in \dot{\mathbb{R}} \times \mathbb{R}_+$ are $\sum_j J_{\Phi_j}(\sum_r x_r^j) = \sum_j \|\sum_r x_r^j\|^2$;
(b) The Jacobian of Φ^{-1}at $u_j \in \mathbb{R}^2\backslash\{(0,0)\}$ are $\sum_j J_{\Phi^{-1}}(u_j) = \sum_j \|\sum_r x_r^j(u_j)\|^{-2} = 1/(2\|u_j\|)$;
(c) $\Phi((a(R)_{\theta_j}\sum_r x_r^j)) = (a^2(R_j)_{2\theta_j}\Phi(\sum_r x_r^j))$ for every a $\in \mathbb{R}$, every $\sum_r x_r^j \in \mathbb{R}^2$;
(d) $(\Phi^{-1}((1+\varepsilon)(R)_{\theta_j}u_j)) = ((1+\varepsilon)^{1/2}(R_j)_{\theta_j/2}\Phi^{-1}(u_j))$, for every $\varepsilon > -1$, every $u_j \in \mathbb{R}^2\backslash\{(0,0)\}$;
(e)$\langle \Sigma^j_{y^j}(\sum_r x_r^j), \sum_r x_r^j\rangle = -2\langle y^j, \Phi(\sum_r x_r^j)\rangle$for every $\sum_r x_r^j \in \dot{\mathbb{R}} \times \mathbb{R}_+$ and every $y^j \in \mathbb{R}^2$.

Proof. First we show that Φdefine a bijective mapping of $\mathbb{R}_+\times \mathbb{R}_+$ onto $\mathbb{R}\times\mathbb{R}_+$. Indeed, for $((\sum_r x_r^j)_1, (\sum_r x_r^j)_2) \in \mathbb{R}_+\times\mathbb{R}_+$ and $(u_j, u_j) \in \mathbb{R}\times\mathbb{R}_+$

$$\begin{cases} \frac{(\sum_r x_r^j)_1^2-(\sum_r x_r^j)_2^2}{2} = (u_j)_1 \\ (\sum_r x_r^j)_1(\sum_r x_r^j)_2 = (u_j)_2 \end{cases} \Leftrightarrow \begin{cases} (\sum_r x_r^j)_1^2 - \frac{(u_j)_2^2}{(\sum_r x_r^j)_1^2} = 2(u_j)_1 \\ (\sum_r x_r^j)_2 = \frac{(u_j)_2}{(\sum_r x_r^j)_1} \end{cases}$$

For fixed $(u_j)_2 \in \mathbb{R}_+$, the map $h_j(1+\varepsilon) = (1+\varepsilon) - (u_j)_2^2/(1+\varepsilon)$ defined in $\mathbb{R}_+$ is increasing since $h'_j(1+\varepsilon) = 1 - (u_j)_2^2/(1+\varepsilon)^2 > 0$. Further, $h_j(1+\varepsilon) \to -\infty$ for $(1+\varepsilon) \to 0^+$ and $h_j(1+\varepsilon) \to +\infty$ for $(1+\varepsilon) \to +\infty$. Therefore, for any given $(u_j)_1 \in \mathbb{R}$ there is exactly one value of $(\sum_r x_r^j)_1^2$ such that $h_j((\sum_r x_r^j)_1^2) = 2(u_j)_1$. Hence, for all given $(u_j)_1 \in \mathbb{R}$ and $(u_j)_2 > 0$ there is a unique $((\sum_r x_r^j)_1, (x^j)_2) \in \mathbb{R}_+\times\mathbb{R}_+$ suchthat $\Phi((\sum_r x_r^j)_1, (\sum_r x_r^j)_2) = ((u_j)_1,(u_j)_2)$. This shows that $\Phi$is bijective from $\mathbb{R}_+\times\mathbb{R}_+$ onto $\mathbb{R}\times\mathbb{R}_-$.

Similarly, it is bijective from $\mathbb{R}_-\times\mathbb{R}_+$onto $\mathbb{R}\times\mathbb{R}_+$and hence from $\dot{\mathbb{R}}\times\mathbb{R}_+$onto$\mathbb{R}^2\backslash\{(0,0)\}$;. It is clearly smooth and its regularity follows from

$$J_\Phi(\sum_r x_r^j) = \det\begin{bmatrix} (\sum_r x_r^j)_1 & -(\sum_r x_r^j)_2 \\ (\sum_r x_r^j)_2 & (\sum_r x_r^j)_1 \end{bmatrix} = (\sum_r x_r^j)_1^2 + (\sum_r x_r^j)_2^2$$

This establishes that $\Phi$is a diffeomorphisms and proves (a). As for (b), it follows from (a)and the observation that

$$(u_j)_1^2 + (u_j)_2^2 = \left(\frac{(\sum_r x_r^j)_1^2-(\sum_r x_r^j)_2^2}{2}\right)^2 + (\sum_r x_r^j)_1^2(\sum_r x_r^j)_2^2 = \frac{1}{4}\left((\sum_r x_r^j)_1^2 + (\sum_r x_r^j)_2^2\right)^2$$

so that $2\sum_j\|u_j\| = \sum_j\|\sum_r x_r^j\|^2$.

(c) Here we compute

$$\Phi\left(a(R)_{\theta_j}\right) = \Phi\left(a(cos\theta_j(\sum_r x_r^j)_1 + sin\,\theta_j(\sum_r x_r^j)_2), a(-sin\theta_j(\sum_r x_r^j)_1 + cos\theta_j(\sum_r x_r^j)_2)\right)$$

$$\Big(\frac{a^2}{2}\Big[\cos 2\theta_j\left(\left(\sum_r x_r^j\right)_1^2 - \left(\sum_r x_r^j\right)_2^2\right) +$$
$$2\sin 2\theta_j\big((\sum_r x_r^j)_1(\sum_r x_r^j)_2\big)\Big], \frac{a^2}{2}\big[2\sin 2\theta_j\big((\sum_r x_r^j)_1^2 - (\sum_r x_r^j)_2^2\big) + \cos 2\theta_j\big((\sum_r x_r^j)_1(\sum_r x_r^j)_2\big)\big]\Big)$$

$$= a^2\begin{bmatrix} \cos 2\theta_j & \sin 2\theta_j \\ -\sin 2\theta_j & \cos 2\theta_j \end{bmatrix}\begin{bmatrix} \frac{(\sum_r x_r^j)_1^2-(\sum_r x_r^j)_2^2}{2} \\ (\sum_r x_r^j)_1(\sum_r x_r^j)_2 \end{bmatrix}$$

$$= a^2(R)_{2\theta_j}\Phi(\sum_r x_r^j)$$

(d) Put a = $(1+\varepsilon)^{1/2}$ and $\psi = 2\theta_j$ in (c) to get $\Phi\left((1+\varepsilon)^{1/2}(R)_{(\psi/2)(\sum_r x_r^j)}\right)(\sum_r x_r^j) = (1+\varepsilon)(R)_\psi \Phi(\sum_r x_r^j)$.
Put next $\Phi(\sum_r x_r^j) = u_j$ and take $\Phi^{-1}$from both sides. This yields $(1+\varepsilon)^{1/2}(R)_{(\psi/2)}\Phi^{-1}(u_j) = \Phi^{-1}\left((1+\varepsilon)(R)_\psi(u_j)\right)$

(e) From the definition of $\Sigma_{y^j}^j$ and of Φ, we obtain

$$\langle \Sigma_{y^j}^j (\textstyle\sum_r x_r^j), \sum_r x_r^j\rangle = \langle \begin{bmatrix} (y^j)_1(\sum_r x_r^j)_1 & (y^j)_2(x^j)_2 \\ (y^j)_2(\sum_r x_r^j)_1 & -(y^j)_1(\sum_r x_r^j)_2 \end{bmatrix}, \begin{bmatrix} (\sum_r x_r^j)_2 \\ (\sum_r x_r^j)_2 \end{bmatrix}\rangle =$$
$$(y^j)_1\left((\textstyle\sum_r x_r^j)_1^2 - (\sum_r x_r^j)_2^2\right) + (y^j)_2\left(2(\sum_r x_r^j)_1(\sum_r x_r^j)_2\right) = 2\langle y^j, \Phi(\sum_r x_r^j)\rangle$$

as desired to conclude the proof.

Theorem 12. The mapping

$$\textstyle\sum_j \mathcal{U}f_j(u_j) = \sum_j \|u_j\|^{-1/2} f_j(\Phi^{-1}(u_j) \qquad u_j \in \dot{\mathbb{R}} \times \mathbb{R}_+$$

defines an isometry of $L^2_{even}(\mathbb{R}^2)$ onto $L^2(\mathbb{R}^2)$ that intertwines π and μ: $\pi(g^j) \circ \mathcal{U} = \mathcal{U} \circ \mu(g^j)$, for every $g^j \in$ SIM(2).

Proof. Let $f_j \in L^2_{even}(\mathbb{R}^2)$. Then, by (b) in Proposition 11

$$\textstyle\sum_j \|\mathcal{U}f_j\|_2^2 = \int_{\mathbb{R}^2} \sum_j |\mathcal{U}f_j(u_j)|^2\, du_j = \int_{\mathbb{R}^2} \frac{1}{\|u_j\|} \sum_j |f_j(\Phi^{-1})|^2\, du_j$$

$$= 2\int_0^{+\infty}\int_{-\infty}^{+\infty} \textstyle\sum_j |f_j(\sum_r x_r^j)|^2 d(\sum_r x_r^j) = \int_{\mathbb{R}^2} \sum_j |f_j(\sum_r x_r^j)|^2 d(\sum_r x_r^j) = \sum_j \|f_j\|_2^2$$

Thus, $\mathcal{U}$ is an isometry. By (31) and (d) in Proposition 11

$$\textstyle\sum_j \pi(1+\varepsilon, y^j, \theta_j)(\mathcal{U}f_j)(u_j) = (1+\varepsilon)\sum_j e^{-2\pi i\langle y^j, u_j\rangle} \mathcal{U}f_j\left((1+\varepsilon)(R)_{\theta_j}(u_j)\right) = \frac{(1+\varepsilon)}{\|(1+\varepsilon)(R)_{-\theta_j}u_j\|^{1/2}} \sum_j e^{-2\pi i\langle y^j, u_j\rangle} f_j\left(\Phi^{-1}\left((1+\varepsilon)(R)_{\theta_j}u_j\right)\right) = \frac{(1+\varepsilon)^{1/2}}{\|(1+\varepsilon)(R)_{\theta_j}(u_j)\|^{1/2}} \sum_j e^{-2\pi i\langle y^j, u_j\rangle} f_j\left((1+\varepsilon)^{1/2}(R)_{\theta_j/2}\Phi^{-1}(u_j)\right)$$

Finally, by (38) and (e) in Proposition 11

$$\textstyle\sum_j \mathcal{U}(\mu(1+\varepsilon, y^j, \theta_j)(f_j))(u_j) = \sum_j \frac{1}{\|u_j\|^{1/2}} (\mu(1+\varepsilon, y^j, \theta_j)(f_j)(\Phi^{-1}(u_j))$$

$$= \textstyle\sum_j \frac{1}{\|u_j\|^{1/2}} (1+\varepsilon)^{1/2} \sum_j e^{-i\pi\langle \Sigma_{y^j}, \Phi^{-1}(u_j), \Phi^{-1}(u_j)\rangle} f_j\left((1+\varepsilon)^{1/2}(R)_{\theta_j/2}\Phi^{-1}(u_j)\right)$$

$$= \textstyle\sum_j \frac{(1+\varepsilon)^{1/2}}{\|u_j\|^{1/2}} e^{-i\pi 2\langle y^j, u_j\rangle} f_j\left((1+\varepsilon)^{1/2} R_{\theta_j/2}\Phi^{-1}(u_j)\right)$$

as desired.

The group TDS(2), the contourlet point of view

We prove Theorem 14. We first explain the connection between TDS(2) and the two-dimensional wavelet theory that leads to the contourlet construction introduced in (Do and Vetterli). The point is that TDS(2) is isomorphic to the group of mappings of (functions on) the plane generated by translations, dilations, and shearing, where the shearing operators are given by

$$(S_{\ell^j} f_j)(\textstyle\sum_r x_r^j) = f_j\left(({}^{(1+\varepsilon)}S_{\ell^j})(\sum_r x_r^j)\right), f_j \in L^2(\mathbb{R})$$

and the matrices S_{ℓ^j} is defined in (20). These are the ingredients of the contourlet frames [10].Just as for curvelets, one allows dilation and translation operations, but the angular selectivity is achieved by a shearing operation rather than a rotation.

Let L denote the two-dimensional subgroup of Sp (2, ℝ) given by

$$L = \left\{\begin{bmatrix} (1+\varepsilon) & 0 \\ -\ell^j(1+\varepsilon) & (1+\varepsilon) \end{bmatrix} : \varepsilon > -1, \ell^j \in \mathbb{R}\right\}$$

The affine action that it induces on $\mathbb{R}^2$leads to the semidirect product H = $\mathbb{R}^2 \rtimes$ L.This action has two open orbits $\mathcal{O}_+$ and $\mathcal{O}_-$in $\mathbb{R}^2$, where $\mathcal{O}_+ = \{((\sum_r x_r^j)_1, (\sum_r x_r^j)_2) : (\sum_r x_r^j)_2 > 0\}$ and $\mathcal{O}_- = \{((\sum_r x_r^j)_1, (\sum_r x_r^j)_2)) \; ; (\sum_r x_r^j)_2 < 0\}$. The wavelet representation ν of H is

$$\nu(1+\varepsilon, y^j, \ell^j) f_j = (T_{y^j} D_{(1+\varepsilon)} S_{\ell^j}) f_j, \;\; f_j \in L^2(\mathbb{R}^2), \tag{40}$$

but it is more convenient to view ν in the frequency domain, namely

$$\pi(1+\varepsilon, y^j, \ell^j) f_j(u_j) = (\mathcal{F} \circ \nu(1+\varepsilon, y^j, \ell^j) f_j)(u_j) = e^{-2\pi i \langle y^j, u_j \rangle} D_{-(1+\varepsilon)} ({}^{(1+\varepsilon)}S_{-\ell^j}) \, f_j\,(u_j) \tag{41}$$

We have $\pi = \pi_{\mathcal{O}_+} \oplus \pi_{\mathcal{O}_-}$, where $\pi_{\mathcal{O}_+}$and $\pi_{\mathcal{O}_-}$ are the subrepresentations of π obtained by restriction to $L^2(\mathcal{O}_+)$ and $L^2(\mathcal{O}_-)$, respectively. The sequence of wavelets ϕ_j such that $\widehat{\phi}_j \in L^2(\mathcal{O}_+)$ are reproducings for $\pi_{\mathcal{O}_+}$ if

$$\int_0^\infty \int_{\mathbb{R}} \sum_j \left| \frac{\widehat{\phi}_j((\sum_r \xi_r^j)_1, (\sum_r \xi_r^j)_2)}{(\sum_r \xi_r^j)_2} \right|^2 (\sum_r \xi_r^j)_1 (\sum_r \xi_r^j)_2 < \infty$$

and similarly for $\pi_{\mathcal{O}_-}$ (see [3] for more details).

If $y^j = ((y^j)_1, (y^j)_2) \in \mathbb{R}^2$, we put $(B_j)_{y^j} = \begin{bmatrix} 0 & (y^j)_1 \\ (y^j)_1 & (y^j)_2 \end{bmatrix}$, a 2 × 2 symmetric matrix. Then, we check that TDS(2) = $G_0 \rtimes G_1$, where

$$G_0 = \left\{ (g^j)_0(1+\varepsilon, y^j) = \begin{bmatrix} (1+\varepsilon)^{-1/2} & 0 \\ (1+\varepsilon)^{-1/2}(B_j)_{y^j} & (1+\varepsilon)^{-1/2} \end{bmatrix} : \varepsilon > -1, y^j \in \mathbb{R}^2 \right\}$$

$$G_1 = \left\{ (g^j)_1(\ell^j) = \begin{bmatrix} S_{\ell^j} & 0 \\ 0 & {}^{(1+\varepsilon)}S_{-\ell^j} \end{bmatrix} : \; \ell^j \in \mathbb{R} \right\}$$

Indeed, for $y^j = ((y^j)_1, (y^j)_2) \in \mathbb{R}^2$, it is easy to see that, using the parameterization in (19)

$$(g^j)_0\,(1+\varepsilon, y^j)(g^j)_1\,(\ell^j) = (A_j)_{(1+\varepsilon), \ell^j, ((y^j)_1, (y^j)_2 + \ell^j (y^j)_1)}$$

so that TDS(2) = $G_0 G_1$ set-theoretically. Furthermore,

$$(g^j)_1\,(\ell^j)(g^j)_0\,(1+\varepsilon, ((y^j)_1, (y^j)_2))(g^j)_1\,(\ell^j)^{-1} = (g^j)_0\,(1+\varepsilon, ((y^j)_1, (y^j)_2 - 2(y^j)_1 \ell^j)). \tag{42}$$

This means that G_1normalizes G_0, so that TDS(2) is the semidirect product $G_0 \rtimes G_1$.Since G_0 is normal in TDS(2), obviously TDS(2) / $G_0 \simeq G_1$. Finally, the products are

$g^j((1+\varepsilon), ((y^j)_1, (y^j)_2), \ell^j) g^j (r, ((z^j)_1, (z^j)_2), s^j) = g^j ((1+\varepsilon)\, r, ((y^j)_1 + (1+\varepsilon)(z^j)_1, (y^j)_2 + (1+\varepsilon)(z^j)_2 - 2\ell^j(1+\varepsilon)(z^j)_1), s^j + \ell^j)$

which implicitly shows the isomorphism TDS(2) ≃ H, as one checks by computing the product in

H =$\mathbb{R}^2 \rtimes$ L. Observe that the decomposition of TDS(2) as a semidirect product is similar to the decomposition of SIM(2) as far as the normal factors are concerned. The basic difference consists in the structure of their quotients: It is compact for SIM(2) and non compact for TDS(2). In order to compute the metaplectic representation on TDS(2), we observe first that the matrix $(A_j)_{(1+\varepsilon), y^j, \ell^j}$in (19) can be written as the product of a diagonal matrix $D_{(1+\varepsilon), \ell^j}$and a lower triangular matrix $L_{(1+\varepsilon), y^j, \ell^j}$as follows

$$(A_j)_{(1+\varepsilon), y^j, \ell^j} = D_{(1+\varepsilon), \ell^j} L_{(1+\varepsilon), y^j, \ell^j} = \begin{bmatrix} (1+\varepsilon)^{-1/2} S_{\ell^j/2} & 0 \\ 0 & (1+\varepsilon)^{1/2} ({}^{(1+\varepsilon)}S_{-\ell^j/2}) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \left(\frac{1}{1+\varepsilon}\right)({}^{(1+\varepsilon)}S_{-\ell^j/2})\,(B_j)_{y^j} S_{\ell^j/2} & 1 \end{bmatrix}$$

We then use the fact that μ is a representation and formulae (6) and (7) to obtain that for $f_j \in L^2\,(\mathbb{R}^2)$

$$\mu\left(A_{j_{(1+\varepsilon), y^j, \ell^j}}\right) f_j(\textstyle\sum_r x_r^j) = \mu\left(D_{(1+\varepsilon), \ell^j} L_{(1+\varepsilon), y^j, \ell^j}\right) f_j(\sum_r x_r^j) = \; (1+\varepsilon)^{1/2} \left(L_{(1+\varepsilon), y^j, \ell^j} f_j\right)((1+\varepsilon)^{1/2} S_{-\ell^j/2}(\sum_r x_r^j))$$

$$= (1+\varepsilon)^{1/2} e^{-i\pi \langle ({}^{(1+\varepsilon)}S_{\ell^j/2})(B_j)_{y^j}(\sum_r x_r^j), S_{-\ell^j/2}(\sum_r x_r^j) \rangle} f_j((1+\varepsilon)^{1/2} S_{-\ell^j/2}(\textstyle\sum_r x_r^j))$$

$$= (1+\varepsilon)^{1/2} e^{-i\pi\langle (B_j)_{y^j} \sum_r x_r^j, \sum_r x_r^j \rangle} f_j\left((1+\varepsilon)^{1/2} S_{-\ell^j/2}(\sum_r x_r^j)\right) \quad (43)$$

The intertwining operator and the equivalence for TDS(2).

We shall be concerned with the mapping

$$\Psi = \dot{\mathbb{R}} \times \mathbb{R}_+ \to \dot{\mathbb{R}} \times \mathbb{R}_+, \quad \sum_r x_r^j \mapsto \left((\sum_r x_r^j)_1 (\sum_r x_r^j)_2, \frac{(\sum_r x_r^j)_2^2}{2}\right), \quad (44)$$

whose properties are described in the following elementary proposition. Its proof is analogous to that of Proposition 11 and is therefore omitted.

Proposition 13. The mapping (44) defines diffeomorphisms from $\dot{\mathbb{R}} \times \mathbb{R}_+$or from

$\dot{\mathbb{R}} \times \mathbb{R}_-$ onto $\dot{\mathbb{R}} \times \mathbb{R}_+$and is such that$\Psi(-\sum_r x_r^j) = \Psi(\sum_r x_r^j)$. Further, it satisfies:

(a) The Jacobian of Ψ at $\sum_r x_r^j = ((\sum_r x_r^j)_1 (\sum_r x_r^j)_2) \in \dot{\mathbb{R}} \times \mathbb{R}_+$ ($\sum_r x_r^j = ((\sum_r x_r^j)_1 (\sum_r x_r^j)_2) \in \dot{\mathbb{R}} \times \mathbb{R}_-$, respectively)
(b)The Jacobian of Ψ^{-1} at $u_j = ((u_j)_1, (u_j)_2) \in \dot{\mathbb{R}} \times \mathbb{R}_+$ is $J_{\psi^{-1}}(u_j) = 1/(2(u_j)_2)$,
(c)$\Psi^{-1}\left((1+\varepsilon)^2 S_{2\ell^j}(u_j)\right) = (1+\varepsilon) S_{\ell^j} \Psi^{-1}(u)$ for every $\varepsilon > -1$ and every $u_j \in \dot{\mathbb{R}} \times \mathbb{R}_+$
(d) By $\langle (B_j)_{y^j} (\sum_r x_r^j), \sum_r x_r^j \rangle = 2\langle y^j, \Psi(\sum_r x_r^j)\rangle$ for every $\sum_r x_r^j \in \dot{\mathbb{R}} \times \mathbb{R}_+$($\sum_r x_r^j =$ $\left((\sum_r x_r^j)_1, (\sum_r x_r^j)_2\right) \in \dot{\mathbb{R}} \times \mathbb{R}_-$respectively) and every $y^j \in \mathbb{R}^2$.

The proof of the following theorem is analogous to the proof of Theorem 12 and its details are given in (Cordero *et al.,* 2005).

Theorem 14. The mapping obtained by extending

$$\mathcal{Q} f_j(u_j) = \left|2(u_j)_2\right|^{-1/2} f_j\left(\Psi^{-1}((u_j)_1, (u_j)_2)\right), \quad u_j \in \dot{\mathbb{R}} \times \mathbb{R}_+$$

to $\dot{\mathbb{R}} \times \dot{\mathbb{R}}$as an even function defines an isometry of L^2_{even} ($\mathbb{R}^2$) onto itself that intertwines the representations π and μ, that are π(g^j) ∘ $\mathcal{Q}$ = $\mathcal{Q}$ ∘ μ(g^j) for every g^j ∈ TDS(2).

Admissible functions for TDS(2).

The reproducibility of TDS(2) follows either by the admissibility condition (16) or directly by the same techniques as in Theorem 17 with the admissibility conditions stated below, (see e.g.,[20]).

Theorem 15. Let H = TDS(2). The identity

$$\int_H \sum_j \left|\langle f_j, \mu(h_j)\phi_j\rangle\right|^2 dh_j = \sum_j c_{\phi_j} \left\|f_j\right\|_2^2 \quad (45)$$

holds for every $f_j \in L^2(\mathbb{R}^2)$ if and only if the sequence of functions ϕ_j satisfies the following two admissibility conditions:

$$\sum_j c_{\phi_j} = 4 \int_{\mathbb{R}} \int_0^\infty \sum_j \left|\phi_j(\sum_r x_r^j)\right|^2 \frac{d(\sum_r x_r^j)_2}{(\sum_r x_r^j)_2^4} \, d(\sum_r x_r^j)_1$$

$$= 4 \int_{\mathbb{R}} \int_0^\infty \sum_j \left|\phi_j(-\sum_r x_r^j)\right|^2 \frac{d(\sum_r x_r^j)_2}{(\sum_r x_r^j)_2^4} \, d(\sum_r x_r^j)_1 \quad (46)$$

and

$$\int_{\mathbb{R}} \int_0^\infty \sum_j \phi_j(\sum_r x_r^j) \overline{\phi_j(-\sum_r x_r^j)} \frac{d(\sum_r x_r^j)_2}{(\sum_r x_r^j)_2^4} \, d(\sum_r x_r^j)_1 = 0 \quad (47)$$

Theorem 14 is proved in (Cordero *et al.,* 2015), where examples of admissible wavelets for TDS(2) are also given.

A class of Reproducing Groups Including SIM(2)

The (double cover of) SIM(2) group is one in a family of reproducing groups parametrized by $\mathbb{R}$. For any parameter pair $(\beta+\varepsilon, \beta) \neq (0, 0)$, consider the 3-dimensional subgroup of Sp(2, $\mathbb{R}$)

$$H_{(\beta+\varepsilon,\beta)} = \begin{bmatrix} e^{-(\beta+\varepsilon)(1+\varepsilon)/2}(R)_{\beta(1+\varepsilon)/2} & 0 \\ \Sigma^{j}_{y^j} e^{-(\beta+\varepsilon)(1+\varepsilon)/2}(R)_{(\beta+\varepsilon)(1+\varepsilon)/2} & e^{(\beta+\varepsilon)(1+\varepsilon)/2}(R)_{(\beta+\varepsilon)(1+\varepsilon)/2} \end{bmatrix} :$$

$1+\varepsilon \in \mathbb{R}, y^j \in \mathbb{R}^2 \subset Sp(2,\mathbb{R})$

where the rotation matrix $(R)_{\theta_j}$ is defined in (29) and the matrix $\Sigma^{j}_{y^j}$in (32). Clearly,

$$h_{(\beta+\varepsilon,\beta)}(1+\varepsilon, y^j) = \begin{bmatrix} I & 0 \\ \Sigma^{j}_{y^j} & 1 \end{bmatrix} \times$$

$$\begin{bmatrix} e^{-(\beta+\varepsilon)(1+\varepsilon)/2}(R)_{\beta(1+\varepsilon)/2} & 0 \\ 0 & e^{(\beta+\varepsilon)(1+\varepsilon)/2}(R)_{\beta(1+\varepsilon)/2} \end{bmatrix}$$

$$= exp\begin{bmatrix} 0 & 0 \\ \Sigma^{j}_{y^j} & 0 \end{bmatrix} exp\left(-\frac{(1+\varepsilon)}{2}\begin{bmatrix} (\beta+\varepsilon)I - \beta I & 0 \\ 0 & -(\beta+\varepsilon)I - \beta J \end{bmatrix}\right)$$

where as usual J = $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Furthermore, $\Sigma^{j}_{y^j} = (y^j)_1 H + (y^j)_2 L$, where H = $\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$. Thus, the Lie algebra $(\mathfrak{h}_j)_{(\beta+\varepsilon)\beta}$ of $H_{(\beta+\varepsilon)\beta}$ is spanned by

$$X = \begin{bmatrix} 0 & 0 \\ H & 0 \end{bmatrix},\ Y = \begin{bmatrix} 0 & 0 \\ L & 0 \end{bmatrix},\ Z = -\frac{1}{2}\begin{bmatrix} (\beta+\varepsilon)I - \beta J & 0 \\ 0 & -(\beta+\varepsilon)I - \beta J \end{bmatrix}$$

Because of the brackets [H, J] = 2L and [L, J] = −2H one sees immediately that

$$\begin{cases} [X,Y] = 0 \\ [X,Y] = (\beta+\varepsilon)X - (1+\varepsilon)Y \\ [X,Y] \ = \beta X + (\beta+\varepsilon)Y \end{cases}$$

According to the classification of three-dimensional Lie algebras (Jacobson, 1979), all $(\mathfrak{h}_j)_{(\beta+\varepsilon,\beta)}$ fall in the class $\mathcal{A}$ = {$\mathfrak{g}_\Gamma$: Γ ∈ GL(2, ℝ)}, where $\mathfrak{g}_\Gamma$= span{X, Y, Z} has bracket table

$$\begin{cases} [X,Y] = 0 \\ [X,Y] = aX + (a+\varepsilon)Y \\ [X,Y] \ = cX + cY \end{cases} \qquad \Gamma = \begin{bmatrix} a & a+\varepsilon \\ c & d \end{bmatrix} \in GL(2,\mathbb{R})$$

In our case

$$\Gamma = \Gamma_{(\beta+\varepsilon,\beta)} = \begin{bmatrix} (\beta+\varepsilon) & -\beta \\ \beta & (\beta+\varepsilon) \end{bmatrix}$$

is nonsingular since det Γ = $(\beta+\varepsilon)^2 + \beta^2 \neq 0$ because $((\beta+\varepsilon), \beta) \neq (0, 0)$. The isomorphism classes in $\mathcal{A}$ are described by Γ, as we now explain. First of all, two multiple matrices Γ and λ Γ give rise to the same algebra if λ≠0, for if $\mathfrak{g}_\Gamma$= span{X, Y, Z}, then the basis {X, Y, λ Z} yields the bracket table that corresponds to λ and generates the same Lie algebra. Thus, $\mathfrak{g}_\Gamma = \mathfrak{g}_{\lambda\Gamma}$ if λ ≠ 0. The isomorphism classes within A_j are in one-to-onecorrespondence with the conjugacy classes in PGL(2, ℝ) = GL(2, ℝ)/($\dot{\mathbb{R}}$ · id). In otherwords, two nonmultiple matrices Γ and Γ́ correspond to isomorphic Lie algebras if andonly if they are conjugate in GL(2, ℝ). It is however an elementary exercise to check that a matrix g^j ∈ GL(2, ℝ) conjugates $\Gamma_{(\beta+\varepsilon,1+\varepsilon)}$into a matrix $\Gamma_{\gamma,\delta}$ of the same type if and only if g^j is multiple of a rotation matrix in SO(2).

In this case, $g^j(\Gamma_{(\beta+\varepsilon),\beta})(g^j)^{-1} = \Gamma_{(\beta+\varepsilon),\beta)}$. Therefore, to each point in

$[(\beta+\varepsilon):\beta] = \{\lambda((\beta+\varepsilon),\beta): \lambda \neq 0\}$in projective space $\mathbb{RP}^2$ there corresponds an isomorphism lass in the subclass $\mathcal{H}$ = $\{(\mathfrak{h}_j)_{(\beta+\varepsilon),\beta}: (\beta+\varepsilon,\beta) \neq (0,0)\}$ of $\mathcal{A}$.

There is another issue that must be discussed, in the light of Theorem 7. One of its consequences is that an admissible subgroup of Sp($1+\varepsilon$, ℝ)cannot be unimodular. This fact is proved in [6] and is really a straightforward adaptation of a theorem proved in [18]. This explains why we have chosen $(\beta+\varepsilon,\beta) \neq (0, 0)$ from the start. Indeed, if $((\beta+\varepsilon),\beta)) = (0, 0)$, then $H_{0,0}$is (two-dimensional and) nilpotent, hence unimodular, and the constructions that follow cannot possibly lead to admissible groups. Furthermore, we must exclude from our parameters all those that correspond to unimodular groups. The modular function Δ on

$H_{(\beta+\varepsilon),\beta}$ is, as for all Lie group, $\Delta(\sum_r x_r^j) = \det(\mathrm{Ad}((\sum_r x_r^j)^{-1})$. If

v = expV is close to the identity,
$\Delta(v) = \det(\mathrm{Ad}(v^{-1})) = \det(\mathrm{Ad}(\exp(-V))) = \det(e^{-ad\,V}) = e^{-(1+\varepsilon)r(ad\,V)}$

shows that $\Delta(v) = 1$ if and only if tr(ad V) = 0. Thus, $H_{(\beta+\varepsilon,\beta)}$ is unimodular if and only if this is true for every $V \in (\mathfrak{h}_j)_{(\beta+\varepsilon),\beta}$. From the bracket table we see that tr(ad Z) = 0 and tr(ad X) = tr(ad Y) = $(\beta + \varepsilon)$. Hence, $H_{(\beta+\varepsilon),\beta}$ is unimodular if and only if $\varepsilon = -\beta$. We summarize this discussion, and other elementary facts, in the proposition that follows.

Proposition 16. The subgroups $H_{(\beta+\varepsilon),\beta}$ of Sp(2,ℝ) satisfy the following properties:

(a) The product law in $H_{(\beta+\varepsilon),\beta}$ is explicitly given by:
$(\mathfrak{h}_j)_{(\beta+\varepsilon),\beta}\,(1+\varepsilon, z^j)(\mathfrak{h}_j)_{(\beta+\varepsilon),\beta}(s^j, z^j) = (\mathfrak{h}_j)_{(\beta+\varepsilon),\beta}((1+\varepsilon) + s^j, y^j + e^{(\beta+\varepsilon)(1+\varepsilon)}(R)_{\beta(1+\varepsilon)}z^j),\ (1+\varepsilon),\ s^j \in \mathbb{R}, y^j, z^j \in \mathbb{R}^2$
(b) The left Haar measure on $H_{(\beta+\varepsilon),\beta}$ are $d(\mathfrak{h}_j)_{(\beta+\varepsilon),\beta}(s^j, z^j) = e^{-2(\beta+\varepsilon)s^j} ds^j\, dz^j$.
(c) $H_{(\beta+\varepsilon),\beta}$ is unimodular if and only if $\varepsilon = -\beta$.
(d) $H_{(\beta+\varepsilon),\beta}$ and $H_{\gamma,\delta}$ are conjugate within Sp(2, ℝ) if and only if they are equal, if and only if $((\beta + \varepsilon), \beta) = \lambda(\gamma, \delta)$, for some $\lambda \neq 0$.
(e) Each $H_{(\beta+\varepsilon),\beta}$ is normalized by the natural copy of SO(2) inside Sp(2, ℝ).
(f) The semidirect product $H_{(1,0)} \rtimes$SO(2) is (canonically) isomorphic to SIM(2).
(g) The restriction of the metaplectic representation to $H_{(\beta+\varepsilon),\beta}$ is given by:

$$\mu((\mathfrak{h}_j)_{(\beta+\varepsilon),\beta}(1+\varepsilon, y^j))f_j(\textstyle\sum_r x_r^j) = e^{(\beta+\varepsilon)(1+\varepsilon)/2} e^{\pi i \langle \sum_{y^j}(\sum_r x_r^j), \sum_r x_r^j\rangle} f_j(e^{(\beta+\varepsilon)(1+\varepsilon)/2}(R)_{-(\beta+\varepsilon)(1+\varepsilon)/2}(\textstyle\sum_r x_r^j)) \quad (48)$$

Proof. The statements follow either from the above discussion or from straightforward computations. We content ourselves with a couple of comments. By the natural copy of SO(2) inside Sp(2, ℝ) we mean of course

$$SO(2) \simeq \left\{(k)_{\theta_j} = \begin{bmatrix}(R)_{\theta_j} & 0 \\ 0 & (R)_{\theta_j}\end{bmatrix} : \theta_j \in [0, 2\pi)\right\} \quad (49)$$

and by (36) one computes immediately $(k)_{\theta_j}(\mathfrak{h}_j)_{(\beta+\varepsilon),\beta}\,(1+\varepsilon, y^j)\,(k)_{\theta_j}^{-1} = (\mathfrak{h}_j)_{(\beta+\varepsilon),\beta}\,(1+\varepsilon, (R)_{2\theta_j}y^j)$, which is the conjugation referred to in (e). As for (f), notice that when $= 1 - \beta$, $\beta = 0$ and $\tau = e^{(1+\varepsilon)}$, the matrix $(\mathfrak{h}_j)_{(1,0)}(\tau, y^j)$ is the G_0 - component of an element in SIM(2).

By (c) and (d), we may assume $\varepsilon = 1 - \beta$, and by (e) we may define the family of groups

$G_\beta = H_{(1,\beta)} \rtimes SO(2)$, $\beta \in \mathbb{R}$.

The elements of G_β will be denote $d\,(g^j)_\beta = (h_j)_\beta\, k$, where k ∈ SO(2). Also, the left Haar measures are $d\,(g^j)_\beta = d(h_j)_\beta\, dk$. In the sequel, we shall parametrize K = SO(2) with the angular parameter θ_j as in (49). We prove next that the groups G_β are all reproducing.

Theorem 17. The identity

$$\int_{G_\beta} \sum_j \left|\langle f_j, \mu(G_\beta)\phi_j\rangle\right|^2 \, dG_\beta = \sum_j c_{\phi_j} \left\|f_j\right\|_2^2 \quad (50)$$

Holds for every $f_j \in L^2(\mathbb{R}^2)$ if and only if the sequence of functions ϕ_j satisfies the following admissibility conditions :

$$\sum_j c_{\phi_j} = 2\int_{\mathbb{R}^2} \sum_j \frac{\left|\phi_j(\sum_r x_r^j)\right|^2}{\left\|\sum_r x_r^j\right\|^4}\, d(\textstyle\sum_r x_r^j) < \infty. \quad (51)$$

$$\int_{\mathbb{R}^2} \sum_j \phi_j\,(\textstyle\sum_r x_r^j)\overline{\phi_j(-\sum_r x_r^j)}\, \frac{d(\sum_r x_r^j)}{\left\|\sum_r x_r^j\right\|^4} = 0 \quad (52)$$

First, we rove an identity of Plancherel type, (see e.g., (Cordero *et al.,* 2005)).
Lemma 18. Let Φ be the mappings defined in (39) and $h_j \in L^2(\mathbb{R}^2)$ be a function which vanishes outside some annulus $c < \sum_j\left\|\sum_r x_r^j\right\| < C$, with $0 < c < C < \infty$. Then

$$\int_{\mathbb{R}^2}\sum_j\left|\int_{\mathbb{R}^2}h_j(\sum_r x_r^j)e^{2\pi i\langle y^j,\phi_j(\sum_r x_r^j)\rangle}d(\sum_r x_r^j)\right|^2 dy^j = \int_0^\infty\int_{\mathbb{R}}\sum_j\left|h_j(\sum_r x_r^j)+h_j(-\sum_r x_r^j)\right|^2\frac{d(\sum_r x_r^j)}{\left\|\sum_r x_r^j\right\|^2}$$

Proof. We make the change of variables $\Phi(\sum_r x_r^j) = u_j$. By (b) in Proposition 11

$$\int_{\mathbb{R}^2}\sum_j h_j(\sum_r x_r^j)e^{2\pi i\langle y^j,\Phi(\sum_r x_r^j)\rangle}d(\sum_r x_r^j) = \int_0^\infty\int_{\mathbb{R}}\sum_j\left(h_j(\sum_r x_r^j)+h_j(-\sum_r x_r^j)\right)e^{2\pi i\langle y^j,\Phi(\sum_r x_r^j)\rangle}d(\sum_r x_r^j)$$

$$\int_{\mathbb{R}^2}\sum_j\left(h_j(\Phi^{-1}(u_j))+h_j(-\Phi^{-1}(u_j))\right)e^{2\pi i\langle y^j,u_j\rangle}d(\sum_r x_r^j)\frac{du_j}{\left\|\sum_r x_r^j(u_j)\right\|^2}$$

By the Plancherel formula we obtain

$$\int_{\mathbb{R}^2}\sum_j\left|\int_{\mathbb{R}^2}\left(h_j(\Phi^{-1}(u_j))+h_j(-\Phi^{-1}(u_j))\right)e^{2\pi i\langle y^j,u_j\rangle}d(\sum_r x_r^j)\frac{du_j}{\left\|\sum_r x_r^j(u_j)\right\|^2}\right|^2 dy^j$$

$$= \int_{\mathbb{R}^2}\sum_j\left|\left(h_j(\Phi^{-1}(u_j))+h_j(-\Phi^{-1}(u_j))\right)\right|^2\frac{du_j}{\left\|\sum_r x_r^j(u_j)\right\|^4} = \int_0^\infty\int_{\mathbb{R}}\sum_j\left|h_j(\sum_r x_r^j)+h_j(-\sum_r x_r^j)\right|^2\frac{d(\sum_r x_r^j)}{\left\|\sum_r x_r^j\right\|^2}$$

As desired.

Corollary 19. Let h_j be as in Lemma 18 Then

$$\int_{\mathbb{R}^2}\sum_j\left|\int_{\mathbb{R}^2}h_j(\sum_r x_r^j)\,e^{2\pi i\langle\Sigma_{y^j}^j(\sum_r x_r^j),\sum_r x_r^j\rangle}\,d(\sum_r x_r^j)\right|^2 dy^j$$

$$= \int_0^\infty\int_{\mathbb{R}}\sum_j\left(\left|h_j(\sum_r x_r^j)\right|^2+\left|h_j(-\sum_r x_r^j)\right|^2+2\mathcal{R}e\,h_j(\sum_r x_r^j)\overline{(-\sum_r x_r^j)}\right)\frac{d(\sum_r x_r^j)}{\left\|\sum_r x_r^j\right\|^2}$$

Proof of Theorem 17. By (48), we must evaluate

$$\int_{H_{(1,\beta)}\rtimes K}\sum_j\left|f_j,\mu((h_j)_\beta k)\,\phi_j)\right|^2 d(h_j)_\beta dk = \int_0^{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\sum_j\left|\int_{\mathbb{R}^2}f_j(\sum_r x_r^j)e^{(1+\varepsilon)/2}e^{2\pi i\langle\Sigma_{y^j}(\sum_r x_r^j),\sum_r x_r^j\rangle}\times\right.$$
$$\left.\overline{\phi_j\left(e^{(1+\varepsilon)/2}(R)_{-(\beta(1+\varepsilon)/2+\theta_j)}\right)d\sum_r x_r^j}\right|^2\times(dy^j e^{-2(1+\varepsilon)}d(1+\varepsilon)d\theta_j). \tag{53}$$

Take f_j as in Lemma 18 and apply Corollary 19 to the right-hand side of (53)

$$\int_{H_{(1,\beta)}\rtimes K}\sum_j\left|f_j,\mu\left(\left((h_j)_\beta k\right)\phi_j\right)\right|^2 d(h_j)_\beta dk = \int_0^{2\pi}\int_{\mathbb{R}}\int_0\int_{\mathbb{R}}^\infty\sum_j\left[\left|f_j(\sum_r x_r^j)\right|^2 e^{(1+\varepsilon)}\left|\phi_j\left(e^{(1+\varepsilon)/2}(R)_{-(\beta(1+\varepsilon)/2+\theta_j)}(\sum_r x_r^j)\right)\right|^2+\right.$$
$$\left.\left|f_j(-\sum_r x_r^j)\right|^2 e^{(1+\varepsilon)}\left|\phi_j\left(-e^{(1+\varepsilon)/2}(R)_{-(\beta(1+\varepsilon)/2+\theta_j)}(\sum_r x_r^j)\right)\right|^2\right]$$

$$+2\mathcal{R}e f_j(\sum_r x_r^j)\overline{f_j(-\sum_r x_r^j)}\,e^{2(1+\varepsilon)}\phi_j$$
$$\left(-e^{(1+\varepsilon)/2}(R)_{-(\beta(1+\varepsilon)/2+\theta_j)}(\sum_r x_r^j)\right)\overline{\phi_j\left(e^{(1+\varepsilon)/2}(R)_{-(\beta(1+\varepsilon)/2+\theta_j)}(\sum_r x_r^j)\right)}$$

$$\times\left(\frac{d(\sum_r x_r^j)}{\left\|\sum_r x_r^j\right\|^2}e^{-2(1+\varepsilon)}\,d(1+\varepsilon)d\theta_j\right) \tag{54}$$

Suppose at first that f_j satisfies the additional properties : $f_j\,((\sum_r x_r^j)_1,(\sum_r x_r^j)_2) = 0$ for $(\sum_r x_r^j)_2 < 0$. Perform now the change of variables given by the mappings

$$(1+\varepsilon),\theta_j)\mapsto e^{(1+\varepsilon)/2}(R)_{-(\beta(1+\varepsilon)/2+\theta_j)}(\sum_r x_r^j) = y^j, \tag{55}$$

a well-defined diffeomorphism. One checks that $d(1+\varepsilon)\,d\theta_j = 2e^{-(1+\varepsilon)}\sum_j\left\|\sum_r x_r^j\right\|^{-2}dy^j$
and hence

$$\int_{H_{(1,\beta)}\rtimes K} \sum_j \left| f_j, \mu\left(\left((h_j)_\beta k\right)\phi_j\right)\right|^2 d(h_j)_\beta dk = \int_0^\infty \int_{\mathbb{R}} \sum_j \left|f_j(\sum_r x_r^{\mathrm{j}})\right|^2 \left(\int_{\mathbb{R}} |\phi_j(y^j)|^2 \frac{2}{\|y^j\|^2} dy^j\right) d(\sum_r x_r^{\mathrm{j}})$$

$$= \sum_j \left\|f_j(\sum_r x_r^{\mathrm{j}})\right\|_2^2 \left(2\int_{\mathbb{R}^2} \frac{|\phi_j(y^j)|^2}{\|y^j\|^2} dy^j\right)$$

If $f_j\ ((\sum_r x_r^{\mathrm{j}})_1, (\sum_r x_r^{\mathrm{j}})_2) = 0$, for $(\sum_r x_r^{\mathrm{j}})_2 > 0$, the same relation holds. This argument shows that if the reproducing formula (50) works for all $f_j \in L^2(\mathbb{R}^{(1+\varepsilon)})$, then it works for f_j vanishings in a half-plane and outside an annulus, so that ϕ_j must fulfil (51). Take now bounded sequence of functions f_j supported in some annulus $c < \sum_j |\sum_r x_r^{\mathrm{j}}| < C$. Then

$$\sum_j G(\theta_j, 1+\varepsilon, \sum_r x_r^{\mathrm{j}}) = 2\sum_j \mathcal{R}e f_j(\sum_r x_r^{\mathrm{j}})\overline{f_{\mathrm{j}}(-\sum_r x_r^{\mathrm{j}})}\ e^{2(1+\varepsilon)}\phi_j\left(-e^{(1+\varepsilon)/2}(R)_{-(\beta(1+\varepsilon)/2+\theta)}(\sum_r x_r^{\mathrm{j}})\right)$$

$$\times\left(\overline{\phi_{\mathrm{j}}\left(e^{(1+\varepsilon)/2}(R)_{-(\beta(1+\varepsilon)/2+\theta)}(\sum_r x_r^{\mathrm{j}})\right)(\sum_r x_r^{\mathrm{j}})}\frac{1}{\left\|\sum_r x_r^{\mathrm{j}}\right\|^2}\right)$$

is integrable with respect to the measures $d(\sum_r x_r^{\mathrm{j}})e^{-2(1+\varepsilon)}\ d(1+\varepsilon)\ d\theta_j$. By performing again the change of variable (55), and using the established values ofc_{ϕ_j}, (54) becomes

$$\int_{H_{(1,\beta)}\rtimes K} \sum_j \left| f_j, \mu\left(\left((h_j)_\beta k\right)\phi_j\right)\right|^2 d(h_j)_\beta dk = \sum_j c_{\phi_j} \left\|f_j\right\|_2^2 + \int_0^{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}^2} \sum_j G(\theta_j, (1+\varepsilon), \sum_r x_r^{\mathrm{j}})\, d(\sum_r x_r^{\mathrm{j}}) e^{-2(1+\varepsilon)} d(1+\varepsilon) d\theta_j$$

The reproducing formula (50) implies that the integrals of $G(\theta_j,\ 1+\varepsilon, \sum_r x_r^{\mathrm{j}})$ vanishes. On the other hand, using once more the change of variable (55)

$$\int_0^{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}^2} \sum_j G(\theta_j, 1+\varepsilon, \sum_r x_r^{\mathrm{j}}) d(\sum_r x_r^{\mathrm{j}}) e^{-2(1+\varepsilon)}\ d(1+\varepsilon) d\theta_j = \int_0^\infty \int_{\mathbb{R}} \sum_j f_j(\sum_r x_r^{\mathrm{j}})\overline{f_{\mathrm{j}}(-\sum_r x_r^{\mathrm{j}})}\, d(\sum_r x_r^{\mathrm{j}}) + \int_{\mathbb{R}} \sum_j \overline{\phi_{\mathrm{j}}(y^{\mathrm{j}})}\,\phi_j(-y^j)\frac{dy^j}{\|y^j\|^4}$$

so that (52) must be true as well.

Conversely, assume that (51) and (52) are satisfied. If f_j are a functions as in Lemma 18, then all the terms (53) are integrable and (50) holds for f_j. We conclude by showing that it actually works for all $f_j \in L^2(\mathbb{R}^2)$. To see this, take $f_j \in L^2(\mathbb{R}^2)$ and let $(f_j)_n$ be a sequences of functions as in Lemma 18 which tends to f_j in the L^2 -norm. Then $F(f_j)_n = \langle (f_j)_n, \mu((g^j)_\beta)\ \phi_j)\rangle$ is a Cauchy sequence on $L^2(G_\beta, d((g^j)_\beta))$which tends pointwise to $F(f_j) = \langle f_j, \mu((g^j)_\beta)\ \phi_j)\rangle$. Since (50) holds for all $(f_j)_n$, it follows that it also holds for f_j.

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