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RESEARCH ARTICLE

DYNAMICS OF STOCHASTIC SIS EPIDEMIC MODEL WITH VERTICAL TRANSMISSION

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ABSTRACT

In this paper, we concerned with dynamic behavior of a stochastic SIS epidemic mrdsodel with vertical transmission. First, the sufficient conditions which can determine the extinction. Then the persistence in mean of the epidemic are presented.

INTRODUCTION

Mathematical models have become essential tools in analyzing and understanding the dynamics of infectious diseases. The SIS epidemic model is an important model in epidemiological patterns and disease control, and it has been studied by many authors (Kermack and Mckendrick, 1927; Gao and Hethcote, 1992; Li and Ma, 2002 and Eldoma, 1999). In real life, some diseases may be passed from one individual to another via vertical transmission. That is to say, vertical transmission of diseases is the passing of an infection to offspring of infected parents. This mode of transmission plays an important role in the spread of disease. In recent years, the studies of epidemic models incorporating vertical transmission have become one of the important areas in mathematical theory of epidemiology (Zhang and Jia, 2014; Ainseba *et al.,* 2016 and Kelatlhegile and Kgosimore, 2016) and they have largely been inspired by the works of Busenberg and Cooke (Busenberg *et al.,* 1983; Busenberg and Cooke, 1988). Some examples of such diseases are AIDS, Hepatitis, Zhaika, etc. To the best of our knowledge, there are only few works on research of SIS epidemic with vertical transmission. Therefore, we propose a deterministic SIS epidemic model with vertical transmission, which reads

$$\begin{cases} S'(t) &= \Lambda + bS(t) - dS(t) - BS(t) + \gamma I(t) - \beta S(t)I(t) + bqI(t), \\ I'(t) &= \beta S(t)I(t) - (d+\alpha)I(t) - BI(t) - \gamma I(t) + bpI(t), \end{cases} \tag{1}$$

where $S(t), I(t)$ denote the number of susceptible individuals and infective individuals at time t respectively. b represents the birth rate, d denotes the death rate, β denotes the average number of adequate contacts with susceptible for an infective individual per unit time, γ denotes the recovery rate, q stands for the probability that a child who is born from infectious mother is susceptible, p stands for the probability that a child who is born from infectious mother is infected. B denotes the output rate. All parameter values are assumed to be nonnegative, $p+q=1$.Denote the total populations $N(t)=S(t)+I(t)$, then

$$\limsup_{t\to+\infty} \frac{\log N(t)}{t} \le \frac{\Lambda}{d+B-b}.$$

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From the biological significant of $N(t)$, throughout this paper, we always assume that $d+B-b>0$. Transfer diagram for model (1) is described in Fig. 1.

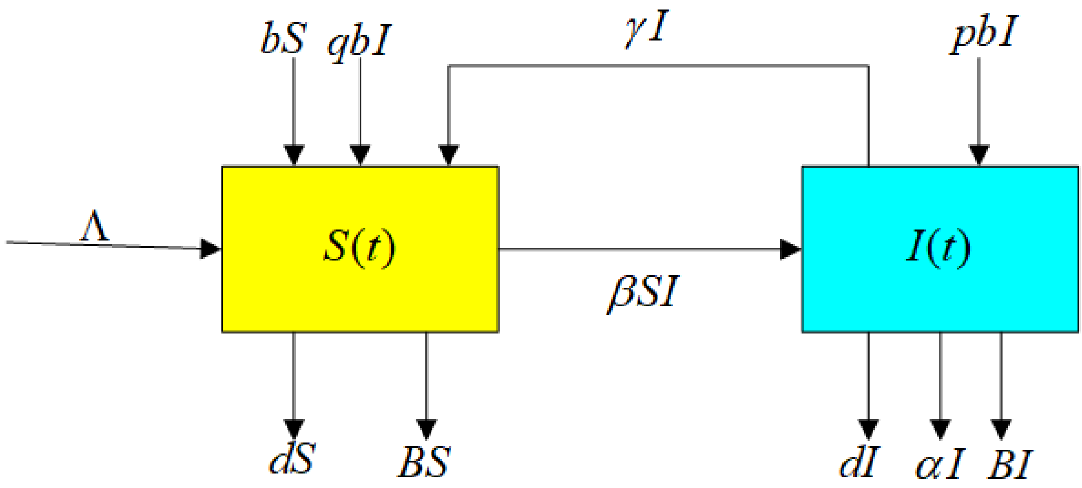

Fig. 1 The transfer diagram for the model (2).

In realism, epidemic models are always affected by the environmental noise (Mao,, 1997). Thus, it is necessary to study how the environmental white noise affects dynamic behavior of the epidemic model. In this article, based on a deterministic epidemic model (1), we shall propose a new model by taking into account the effect of white noise. We assume that fluctuations in the environment will manifest themselves mainly as fluctuations in the parameter β, that is $\beta dt+\sigma dB(t)$. Then we get the stochastic analogue of system (1):

$$\begin{cases} dS(t) &= (\Lambda-(d+B-b)S+\gamma I-\beta SI+bqI)dt-\sigma SIdB(t), \\ dI(t) &= (\beta SI+bpI-(d+\alpha+B+\gamma)I)dt+\sigma SIdB(t), \end{cases} \qquad (2)$$

where σ is a positive coefficient and $B(t)$ is a standard Brownian motion.

The rest of this paper is organized as follows. In section 2, we deduce the condition which will lead the disease to die out. The condition for the disease being persistent in mean is given in next section.

Extinction

Theorem 1.1 For any given initial value $(S(0),I(0))\in R_+^2$, there is a unique global positive solution $(S(t),I(t))$ of system (1) for all $t\geq 0$, and the solution will remain in R_+^2 with probability 1. The proof of this theorem is rather standard and hence is omitted.

Theorem 1.2 Let $(S(t),I(t))$ be the solution of system (2) with initial value $(S(0),I(0))\in\Gamma^*$.

If $\sigma^2\leq\dfrac{\beta(d+B-b)}{\Lambda}$ and $R_0<1$

$$\limsup_{t\to+\infty}\frac{\log I(t)}{t}\leq(d+B+\alpha+\gamma-bp)(\tilde{R}_0-1)<0 \text{ or}$$

if $\sigma^2>\dfrac{\beta(d+B-b)}{\Lambda}\vee\dfrac{\beta^2}{2(d+B-b+\alpha+\gamma)}$

$$\limsup_{t\to+\infty}\frac{\log I(t)}{t}\leq\frac{\beta^2}{2\sigma^2}-(d+B+\alpha+\gamma-bp)<0.$$

namely $I(t)$ tends to zero exponentially with probability one.

Proof: For the system (2), we have

$$\frac{d(S+I)}{dt}=\Lambda-(d+B-b)S-(d+B-b+\alpha)I, \tag{3}$$

so

$$\overline{S}(t)=\frac{\Lambda}{d+B-b}-\frac{d+B-b+\alpha}{d+B-b}\overline{I}(t)-\varphi(t),$$

where $\varphi(t)=\frac{1}{d+B-b}\left(\frac{S(t)+I(t)}{t}-\frac{S(0)+I(0)}{t}\right)$ and $\lim_{t\to+\infty}\varphi(t)=0$ a.s.

By the Ito formula, we have

$$d(\log I(t))=\left(\beta S(t)-(d-bp+\alpha+\gamma+B)-\frac{\sigma^2}{2}S^2(t)\right)+\sigma S(t)dB(t),$$

Then

$$\begin{aligned}\frac{\log I(t)}{t} &=\frac{\log I(0)}{t}+\beta\overline{S}(t)-(d-bp+B+\alpha+\gamma)-\frac{\sigma^2}{2}\frac{1}{t}\int_0^t S^2(r)dr\\ &+\frac{\sigma}{t}\int_0^t S(r)dB(r)\\ &\le\frac{\log I(0)}{t}+\beta\overline{S}(t)-(d-bp+B+\alpha+\gamma)-\frac{\sigma^2}{2}(\overline{S}(t))^2+\frac{\sigma}{t}\int_0^t S(r)dB(r)\\ &=\frac{\log I(0)}{t}+\beta\left(\frac{\Lambda}{d+B-b}-\frac{d+B-b+\alpha}{d+B-b}\overline{I}(t)-\varphi(t)\right)-(d-bp+B+\alpha+\gamma)\\ &-\frac{\sigma^2}{2}\left(\frac{\Lambda}{d+B-b}-\frac{d+B-b+\alpha}{d+B-b}\overline{I}(t)-\varphi(t)\right)^2+\frac{\sigma}{t}\int_0^t S(r)dB(r)\\ &=\frac{\beta\Lambda}{d+B-b}-\left(d-bp+B+\alpha+\gamma+\frac{\sigma^2\Lambda^2}{2(d+B-b)^2}\right)-\frac{\sigma^2}{2}\left(\frac{d+B-b+\alpha}{d+B-b}\right)^2\\ &\times(\overline{I}(t))^2-\left(\frac{(d+B-b+\alpha)(\beta(d+B-b)-\sigma^2\Lambda)}{(d+B-b)^2}\right)\overline{I}(t)+\phi(t),\end{aligned} \tag{4}$$

where

$$\begin{aligned}\phi(t)&=\frac{\log I(0)}{t}-\left(\beta-\frac{\sigma^2\Lambda}{d+B-b}\right)\varphi(t)-\frac{\sigma^2(d+B-b+\alpha)\varphi(t)}{d+B-b}\times\overline{I}(t)\\ &-\frac{\sigma^2}{2}\varphi^2(t)+\frac{\sigma}{t}\int_0^t S(r)dB(r)\le\frac{\log I(0)}{t}+\left(\beta+\frac{\sigma^2\Lambda}{d+B-b}+\frac{\sigma^2(d+B-b+\alpha)}{d+B-b}\right)|\varphi(t)|\\ &-\frac{\sigma^2}{2}\varphi^2(t)+\frac{\sigma}{t}\int_0^t S(r)dB(r),\end{aligned}$$

And

$\phi(t)\ge\frac{\log I(0)}{t}-\left(\beta+\frac{\sigma^2\Lambda}{d+B-b}+\frac{\sigma^2(d+B-b+\alpha)}{d+B-b}\right)|\varphi(t)|-\frac{\sigma^2}{2}\varphi^2(t)+\frac{\sigma}{t}\int_0^t S(r)dB(r).$ Since $\lim_{t\to+\infty}\varphi(t)=0$ a.s. and $\lim_{t\to+\infty}\frac{1}{t}\int_0^t S(r)dB(r)=0$ a.s. according to strong law of large numbers, we have $\lim_{t\to+\infty}\phi(t)=0$ a.s.

Case 1: If $\sigma^2 \le \frac{\beta(d+B-b)}{\Lambda}$, then it follows from (4) that

$$\frac{\log I(t)}{t} \le \frac{\beta\Lambda}{d+B-b} - \left(d - bp + B + \alpha + \gamma + \frac{\sigma^2\Lambda^2}{2(d+B-b)^2}\right) + \phi(t),$$

which together with the property of $\phi(t)$ implies

$$\begin{aligned}\limsup_{t\to+\infty}\frac{\log I(t)}{t} &\le \frac{\beta\Lambda}{d+B-b} - \left(d - bp + B + \alpha + \gamma + \frac{\sigma^2\Lambda^2}{2(d+B-b)^2}\right)\\ &= (d - bp + B + \alpha + \gamma)(\tilde{R}_0 - 1) < 0\end{aligned}$$

Case 2: If $\sigma^2 > \frac{\beta(d+B-b)}{\Lambda} \vee \frac{\beta^2}{2(d+B-b+\alpha+\gamma)}$, then

$$\begin{aligned}\frac{\log I(t)}{t} &\le \frac{\beta\Lambda}{d+B-b} - \left(d - bp + B + \alpha + \gamma + \frac{\sigma^2\Lambda^2}{2(d+B-b)^2}\right)\\ &\quad - \left(\frac{(d+B-b+\alpha)(\beta(d+B-b)-\sigma^2\Lambda)}{(d+B-b)^2}\right)\overline{I}(t) - \frac{\sigma^2}{2}\left(\frac{d+B-b+\alpha}{d+B-b}\right)^2\left(\overline{I}(t)\right)^2 + \phi(t)\end{aligned}$$

$$\begin{aligned}&\le \frac{\beta\Lambda}{d+B-b} - \left(d - bp + B + \alpha + \gamma + \frac{\sigma^2\Lambda^2}{2(d+B-b)^2}\right) + \frac{(\sigma^2\Lambda - \beta(d+B-b))^2}{2\sigma^2(d+B-b)^2} + \phi(t)\\ &= \frac{\beta^2}{2\sigma^2} - (d - bp + B + \alpha + \gamma) + \phi(t),\end{aligned}$$

Therefore

$$\limsup_{t\to+\infty}\frac{\log I(t)}{t} \le \frac{\beta^2}{2\sigma^2} - (d - bp + B + \alpha + \gamma) < 0 a.s.$$

Remark 1.3 From theorem 1.2 we can obtain $\lim_{t\to+\infty} I(t) = 0$ a.s.

For (3),

$$\frac{d(S(t)+I(t))}{dt} = \Lambda - (d+B-b)(S(t)+I(t)) - \alpha I(t),$$

then

$$S(t)+I(t) = e^{-(d+B-b)t}\left[\left(S(0)+I(0)\right) + \int_0^t (\Lambda - \alpha I(r))e^{(d+B-b)r}dr\right].$$

Applying L'Hospital's rule leads to

$$\lim_{t\to+\infty}(S(t)+I(t)) = \frac{\Lambda}{d+B-b} \text{ a.s.}$$

Therefore, we derive

$$\lim_{t\to+\infty} S(t) = \frac{\Lambda}{d+B-b} \text{ a.s.}$$

Persistence of the disease

Theorem 3.1 If

$\tilde{R}_0 > 1$ and $\sigma^2 \le \frac{\beta(d+B-b)}{\Lambda}$, then for any initial value $(S(0), I(0)) \in R_+$, the solution of the system (2) obeys

$\liminf_{t\to+\infty} \overline{I}(t) \ge \xi_1$ a.s.

and

$\limsup_{t\to+\infty} \overline{I}(t) \le \xi_2$ a.s.

Where

$$\xi_1 = \frac{(\tilde{R}_0 - 1)(d+B-b)(d-bp+B+\alpha+\gamma)}{\beta(d+B-b+\alpha)}, \ \xi_2 = \frac{(\tilde{R}_0 - 1)(d+B-b)^2(d-bp+B+\alpha+\gamma)}{(\beta(d+B-b)-\sigma^2\Lambda)(d+B-b+\alpha)}.$$

That is, $I(t)$ will rise to infinitely often with probability one.

Proof : If $\tilde{R}_0 > 1$ and $\sigma^2 \le \frac{\beta(d+B-b)}{\Lambda}$ then from (3) we have

$$\frac{\log I(t)}{t} \le \frac{\beta\Lambda}{d+B-b} - \left(d-bp+B+\alpha+\gamma+\frac{\sigma^2\Lambda^2}{2(d+B-b)^2}\right)$$
$$-\left(\frac{(d+B-b+\alpha)(\beta(d+B-b)-\sigma^2\Lambda)}{(d+B-b)^2}\right)\overline{I}(t)+\phi(t)$$
$$=(d-bp+B+\alpha+\gamma)(\tilde{R}_0-1)-\left(\frac{(d+B-b+\alpha)(\beta(d+B-b)-\sigma^2\Lambda)}{(d+B-b)^2}\right)$$
$$\times\overline{I}(t)+\phi(t),$$

which together with Lemma (11), we have

$$\limsup_{t\to+\infty} \overline{I}(t) \le \frac{(\tilde{R}_0 - 1)(d+B-b)^2(d-bp+B+\alpha+\gamma)}{(\beta(d+B-b)-\sigma^2\Lambda)(d+B-b+\alpha)} \sim\text{a.s.}$$

On the other hand,

$$\frac{\log I(t)}{t} = \frac{\log I(0)}{t} + \beta\overline{S}(t) - (d-bp+B+\alpha+\gamma) - \frac{\sigma^2}{2}\frac{1}{t}\int_0^t S^2(r)dr + \frac{\sigma}{t}\int_0^t S(r)dB(r)$$

$$\ge \frac{\log I(0)}{t} + \beta\left(\frac{\Lambda}{d+B-b} - \frac{d+B-b+\alpha}{d+B-b}\overline{I}(t) - \varphi(t)\right) - (d-bp+B+\alpha+\gamma)$$
$$-\frac{\sigma^2\Lambda^2}{2(d+B-b)^2} + \frac{\sigma}{t} int_0^t S(r)dB(r)$$

$$= \frac{\beta\Lambda}{d+B-b} - \left(d-bp+B+\alpha+\gamma+\frac{\sigma^2\Lambda^2}{2(d+B-b)^2}\right) - \frac{\beta(d+B-b+\alpha)}{(d+B-b)}\overline{I}(t) + \psi(t)$$
$$=(d-bp+B+\alpha+\gamma)(\tilde{R}_0-1) - \frac{\beta(d+B-b+\alpha)}{(d+B-b)}\overline{I}(t)+\psi(t),$$

where $\psi(t) = \frac{\log I(0)}{t} - \beta\varphi(t) + \frac{\sigma}{t}\int_0^t S(r)dB(r)$ and $\limsup_{t\to+\infty}\psi(t) = 0$ a.s. then together with Lemma (11), we have

$$\liminf_{t\to+\infty} \overline{I}(t) \geq \frac{(\tilde{R}_0 - 1)(d+B-b)(d-bp+B+\alpha+\gamma)}{\beta(d+B-b+\alpha)} \quad \text{a.s.}$$

Remark 3.2 Theorem 3.1 tells us that the infected population is persistent under some conditions, we can prove the susceptible population $S(t)$ is also persistence in this situation. In fact, from (3) we get

$$\overline{S}(t) = \frac{\Lambda}{d+B-b} - \frac{d+B-b+\alpha}{d+B-b}\overline{I}(t) - \frac{(S(t)+I(t))-(S(0)+I(0))}{(d+B-b)t}.$$

Furthermore, $\liminf_{t\to+\infty} \overline{I}(t) \geq \xi_1$ indicates that for $\forall \eta > 0 (\eta < \xi_1)$, there exists a $T(\omega)$ such that $\overline{I}(t) \geq \xi_1 - \eta,$ for $t \geq T(\omega)$. Then we get

$$\overline{S}(t) \leq \frac{\Lambda}{d+B-b} - \frac{d+B-b+\alpha}{d+B-b}(\xi_1 - \eta) - \frac{(S(t)+I(t))-(S(0)+I(0))}{(d+B-b)t}.$$

Letting $t \to \infty$ and η arbitrary, we get

$$\limsup_{t\to+\infty} \overline{S}(t) \leq \frac{\Lambda}{d+B-b} - \frac{(\tilde{R}_0 - 1)(d-bp+B+\alpha+\gamma)}{\beta} \text{~a.s.}$$

Note that $\limsup_{t\to+\infty} \overline{I}(t) \leq \xi_2$, then by similar arguments, we have

$$\liminf_{t\to+\infty} \overline{S}(t) \geq \frac{\Lambda}{d+B-b} - \frac{(d-bp+B+\alpha+\gamma)(\tilde{R}_0 - 1)(d+B-b)}{\beta(d+B-b) - \sigma^2\Lambda} \text{ a.s.}$$

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