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RESEARCH ARTICLE

IDEMPOTENT SEPARATING FUZZY CONGUENCES

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ABSTRACT

This work mainly focuses on the developmental study of special classes of fuzzy relations. The discussion of fuzzy orthodox semigroups, as a special class of fuzzy relations, the combination and rearrangement of the idempotent elements of any concerned subject are significant in this work. So the extension of the study of fuzzy Green's relations (G.Hariprakash, 2016), to the area of fuzzy generalized inverse semigroup becomes more relevant. As a part and the course of development, this work, consider equivalence relations introduced by Yamada (M. Yamada, 1967) and the set of fuzzy congruences on semigroups. For more descriptive study of the subject along with the defined descriptions (G.Hariprakash, 2016), (G.Hariprakash,2016),(G.Hariprakash, 2017) a necessary and sufficient condition of idempotent separating fuzzy congruences, and a new notion called strictly idempotent separating fuzzy congruences is defined. Howie found out the greatest idempotent separating congruences on an inverse semigroup(J.M Howie,1976) and Kuroki introduced idempotent separating fuzzy congruences on inverse semigroups. These results are used to define idempotent separating fuzzy congruences fuzzy Green's relations on an inverse semigroup (] J.P. Kim and D.R. Bae, 1997). Moreover, this work extends the concept of group fuzzy congruences to Green's fuzzy relations.

INTRODUCTION

Definition 1.1 *Fuzzy relations*. Let S be a semigroup. A real valued function defined from S×S to [0,1] is called a fuzzy relation (Adlasnig, 1986), (Al-thukair, 1998).

Definition 1.2 *Dot composition of fuzzy relation*

Let α and β be two fuzzy binary relations on a semigroup S. Then the dot composition of fuzzy relations is denoted by $\alpha \circ \beta$ and is defined as

$$\alpha \circ \beta (x, y) = \max_{z \in S} \min \{\alpha(x, z), \beta(z, y)\} \ \forall x,y \in S.$$ [Clifford and Preston, 1967]

Definition 1.3 Similarity relation

A fuzzy binary relation μ defined on a semigroup is a similarity relation if

(i) $\mu (x, x) = 1 \ \forall x \in S$ (reflexive)

(ii) $\mu (x, y) = \mu (y, x)$ (symmetric)

(iii) $\mu \circ \mu \leq \mu$ (transitive)

Definition 1.4 Compatible fuzzy relations

A fuzzy relation μ on a semigroup S is compatible [Hariprakash, 2016] if

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(i) μ (a, b) ≤ μ (a, tb) and μ (a, b) ≤ μ (ta, b) ∀a,b, t ∈S(left compatible)

(ii)) μ (a, b) ≤ μ (a,bt) and μ (a, b) ≤ μ (a, bt) ∀a,b, t ∈S(right compatible)

Definition 1.5 Fuzzy congruence relation

A fuzzy compatible similarity relation on a semigroup is called a fuzzy congruence relation (Hariprakash, 2017), (Kim and Bae, 1997).

2. Idempotent separating fuzzy congruences

As introduced by Yamada (Yamada, 1967) for the case of a generalized inverse semigroup we consider the equivalence relation

$\gamma_L = \{ \hat{\mathcal{L}}_x, L_y) \in S/\hat{\mathcal{L}} \times S/\hat{\mathcal{L}} : V(\hat{\mathcal{L}}x) = V(\hat{\mathcal{L}}y)\}$

$\gamma_R = \{ \hat{\mathcal{R}}_x, \hat{\mathcal{R}}_y) \in S/\hat{\mathcal{R}} \times S/\hat{\mathcal{R}} : V(\hat{\mathcal{R}}_x) = V(\hat{\mathcal{R}}_y)\}$

In this study con_f(S) denotes the set of all fuzzy congruences on a semigroup S.

Definition 2.1 An element μ∈ Con_f(S) is said to be idempotent separating if

$\mu_e = \mu_f \Rightarrow e = f$. In particular (e, f) ∈μ ⇒ e = *f* where e, f ∈ E_s.

Definition 2.2. An element μ∈Con_f(s) is said to be strictly idempotent separating if *(e,f) μ∈ ⇔ e = f where* e, *f* ∈E_s. In particular, $\mu_e = \mu_e = \mu_f \Leftrightarrow e = f$.

Lemma 2.3 (Theorem 3.2) *If S is an inverse semigroup with semilattice of idempotent E_s, the relation.* [9]

$\eta = \{(a,b) \in S \times S : a^{-1}ea = b^{-1}eb \ \forall \in E_s\}$ is the greatest idempotent separating congruence on S.

Lemma 2.4 (Proposition 2.7) *Let μ be a fuzzy congruence relation on an inverse semigroup S. Then S/μ is an inverse semigroup and* $\mu\, [a^{-1}, b^{-1}) = \mu\, (a,b)\ \forall a,b \in S$. (Al-thukair, 1998). Lemma 2.5 (Theorem 4.2) *Let S be an in- inverse semigroup. Then the characteristic function of η on S denoted by λ_η is an idempotent separating fuzzy congruence on S.* (Kuroki, 1997).

Remark 2.6. Fuzzy congruence Green's relations $\hat{\mathcal{L}}$ and $\hat{\mathcal{R}}$ on an inverse semigroup *S* are idempotent separating fuzzy congruences when $\hat{\mathcal{L}}_e = \hat{\mathcal{L}}_f \Rightarrow e = f \Rightarrow e = f$ and $\hat{\mathcal{R}}_e = \hat{\mathcal{R}}_f \Rightarrow e = f$

Proposition 2.7. *If μ is a fuzzy congruence (similarity relation) relation on a commutative semigroup S, then* $\hat{\mathcal{L}}^{-1}(1) = \hat{\mathcal{L}}$ *and* $\hat{\mathcal{R}}^{-1}(1) = \hat{\mathcal{R}}$.

Proof. By the definition of $\hat{\mathcal{L}}^{-1}(1)$, $\hat{\mathcal{R}}^{-1}$ …...(1)

$\hat{\mathcal{L}}^{-1}(1) \le \hat{\mathcal{L}}$ ......(3)

Suppose $(a, b) \in \hat{\mathcal{L}} \Rightarrow \mu_a = \mu_b$. Then μ (*a, b*) = 1.
Since μ is a fuzzy congruence it is symmetric. So, μ (*b,a*) = 1. Then we have

$$\hat{\mathcal{L}}(a,b) \ge \max_{z\in S} \min \{\mu(a, z), \mu(b,z)\}$$
$$\ge \min \{\mu(a,b), \mu(b, a)\}$$
$$\ge \min \{1, 1\}$$
$$\ge 1.$$

Hence $\hat{\mathcal{L}}$(1); So,

$\hat{\mathcal{L}} \le \hat{\mathcal{L}}^{-1}(1)$. ……(4)

From (3) and (4), $\hat{\mathcal{L}} = \hat{\mathcal{L}}^{-1}(1)$. Hence the result.

Similarly, we can prove that $\hat{\mathcal{R}}^{-1}(1) = \hat{\mathcal{R}}$, when μ is a fuzzy congruence,

Theorem 2.8. Let μ be a similarity relation on a commutative inverse semigroup S. Then fuzzy congruence Green's relation $\hat{\mathcal{L}}$ is an idempotent separating fuzzy congruence on *S* if and only if $\hat{\mathcal{L}} \leq \eta$.

Proof. Given μ is a similarity relation on *S* and $\hat{\mathcal{L}} \in Con f(s)$. Assume $\hat{\mathcal{L}}$ is an idempotent separating fuzzy congruence on *S*. Let $(a,b) \in \hat{\mathcal{L}}$. Then by proposition 2.7 $\hat{\mathcal{L}} = \hat{\mathcal{L}}^{-1}(1)$. So $(a, b) \in \hat{\mathcal{L}}^{-1}(1)$.

Then $\hat{\mathcal{L}}(a,b)= 1$, since $\hat{\mathcal{L}}$ is a fuzzy congruence on S by lemma2.4,

$\hat{\mathcal{L}}(a^{-1}, b^{-1}) = \hat{\mathcal{L}}(a,b)= 1$

Now, we have

$$\begin{aligned}\hat{\mathcal{L}}(a^{-1}ea, b^{-1}eb) &\geq \hat{\mathcal{L}} \circ \hat{\mathcal{L}}(a^{-1}ea, b^{-1}eb)\\ &\geq \max_{z\in S} \min \{\hat{\mathcal{L}}(a^{-1}ea,z), \hat{\mathcal{L}}(z, b^{-1}eb)\}\\ &\geq \min \{\hat{\mathcal{L}}(a^{-1}ea,b^{-1}ea), \hat{\mathcal{L}}(b^{-1}ea,b^{-1}eb)\}, \text{ for } z = b^{-1}ea \in S.\\ &\geq \min \{\hat{\mathcal{L}}(a^{-1},b^{-1}), \hat{\mathcal{L}}(a, b)\}, \text{ since } \hat{\mathcal{L}} \text{ is fuzzy compatible}\\ &\geq \min \{1,1\}.\end{aligned}$$

Therefore $\hat{\mathcal{L}}(a^{-1}ea,b^{-1}eb)= 1 \Rightarrow \hat{\mathcal{L}}_{a-1ea}= \hat{\mathcal{L}}_{b-1eb}$.

Since $\hat{\mathcal{L}}$ is idempotent separating fuzzy congruence on *S*,

$\hat{\mathcal{L}}_{a-1ea} = \hat{\mathcal{L}}_{b-1eb} \Rightarrow a^{-1} ea = b^{-1} eb$.

where $e \in E_s$. Then by lemma 4.4.3, $(a, b) \in \eta$. Hence, $(a, b) \in \hat{\mathcal{L}} \Rightarrow$
$(a, b) \in \eta$.

Hence, $\hat{\mathcal{L}} \leq \eta$.

Conversely, assume $\hat{\mathcal{L}} \leq \eta$. That is,

$\hat{\mathcal{L}}^{-1}(1) \leq \eta$. ……(4.9)

Consider $e, f \leq E_s$ such that $(e, f) \in \hat{\mathcal{L}}$. From (4.9) $(e, f) \in \eta$. Since η itself is an idempotent separating fuzzy congruence, and $(e, f) \in \eta \Rightarrow e = f$. Hence, $(e, f) \in \hat{\mathcal{L}} \Rightarrow e = f$.

Therefore, $\hat{\mathcal{L}}$ is an idempotent separating fuzzy congruence on *S*.

Theorem 2.9. Let μ be a similarity relation on a commutative inverse semigroup *S*. Then fuzzy congruence Green's relation $\hat{\mathcal{R}}$ is idempotent separating fuzzy congruence on *S* if and only if and only $\hat{\mathcal{R}} \leq \eta$.

Proof. The result follows from theorem 2.8 by using the property of $\hat{\mathcal{R}}$.

3. Group fuzzy Congruences on Inverse Semigroups

Definition 3.1. Let δ be a fuzzy congruence on a semigroup *S*. If *S*/δ is a group, δ is called group fuzzy congruence.

Proposition 3.2. *Let S be an inverse semigroup and δ is a fuzzy congruence on S. Then δ is a group fuzzy congruence if and only if* $\delta_e = \delta f\ \forall e, \in E_s$.

Proof. Given S is an inverse semigroup. By lemma 2.4. S/δ is an inverse semigroup. Then S/δ is a group if and only if it has only one idempotent. That is, S/δ is a group if and only if $\delta_e = \delta f\ \forall e, \in E_s$.

That is, δ is a group fuzzy congruence if and only if $\delta_e = \delta f\ \forall e, f \in E_s$.

Lemma 3.3 (Theorem 3.1*) If S is an inverse semigroup with semilattice of idempotent E_s, the relation.*

$\sigma = \{(a, b) \in S \times S : ea = eb \text{ for some } e \in E_s\}$

is the least group congruence relation on S. (Howie,1976).

From the above lemma, N Kuroki introduced following lemma.

Lemma 3.4 (Theorem 5.1) *Let S be an inverse semigroup. Then λ_σ is a group fuzzy congruence relation on S where λ_σ is the characteristic function of σ on $S/\hat{\mathcal{L}}$* (Kuroki, 1997).

Theorem 3.5 *If $\hat{\mathcal{L}}$ is a fuzzy congruence Green's relation on an inverse semigroup S, and $E_{S/\hat{\mathcal{L}}}$ the semilattice of idempotents in $S/\hat{\mathcal{L}}$, the relation*

$\sigma_l = \{\hat{\mathcal{L}}_a, \hat{\mathcal{L}}_b) \in S/\hat{\mathcal{L}} \times S/\hat{\mathcal{L}} : \hat{\mathcal{L}}_{ea} = \hat{\mathcal{L}}_{eb}$ for some $e \in E_s\}$ is the least group fuzzy congruence relation on $S/.\hat{\mathcal{L}}$

Proof. Given, $\hat{\mathcal{L}}$ is a fuzzy congruence. Then by lemma 2.4, $S/\hat{\mathcal{L}}$ is an inverse semigroup. By hypothesis, $E_{S/\hat{\mathcal{L}}}$ is the semilattice of idempotents in $S/\hat{\mathcal{L}}$. That is $S/\hat{\mathcal{L}}$, an inverse semigroup with semilattice of idempotents $E_{S/\hat{\mathcal{L}}}$, then by lemma 3.3.
$\sigma_{\hat{\mathcal{L}}} = \{(\hat{\mathcal{L}}_a, \hat{\mathcal{L}}_b) : \hat{\mathcal{L}}_{ea} = \hat{\mathcal{L}}_{eb}$ for some $e \in E_s\}$
is the least group fuzzy congruence relation on $S/\hat{\mathcal{L}}$.

Theorem 3.6. *If $\hat{\mathcal{R}}$ is a fuzzy congruence Green's relation on a semigroup S, and $E_{S/\hat{\mathcal{R}}}$, the semilattice of idempotents in $S/\hat{\mathcal{R}}$, the fuzzy relation $\sigma_{\hat{\mathcal{L}}} = \{(\hat{\mathcal{R}}_a, \hat{\mathcal{R}}_b) \in S/\hat{\mathcal{R}} \times S/\hat{\mathcal{R}} : \hat{\mathcal{R}}_{ea} = \hat{\mathcal{R}}_{eb}$ for some $e \in E_s\}$ is the least group fuzzy congruence relation on $S/\hat{\mathcal{R}}$.*

Proof. Given $\hat{\mathcal{R}}$ is a fuzzy congruence. Then $S/\hat{\mathcal{R}}$ is an inverse semigroup. That is, $S/\hat{\mathcal{R}}$ is an inverse semigroup with semilattice of idempotents $E_{S/\hat{\mathcal{R}}}$ and $\sigma_{\hat{\mathcal{L}}} = \{(\hat{\mathcal{R}}_a, \hat{\mathcal{R}}_b) \in S/\hat{\mathcal{R}} \times S/\hat{\mathcal{R}} : \hat{\mathcal{R}}_{ea} = \hat{\mathcal{R}}_{eb}\}$ so that the result follows from lemma 3.3.

Note 3.7

If S is an inverse semigroup, E_s is commutative and for $e\,f\,e \in E_s \Rightarrow (e\,f\,e)e = (e\,f\,e)\,f$ [2] (Theorem 1.2).
Theorem 3.8. *Let S be an inverse semigroup. $\hat{\mathcal{L}}$ is a fuzzy congruence Green's relation on S and $\mu_{\hat{\mathcal{L}}}$ is a fuzzy congruence on $S/\hat{\mathcal{L}}$. Then the fuzzy relation $\mu_{\sigma\hat{\mathcal{L}}}$ is a group fuzzy congruence on $S/\hat{\mathcal{L}}$.*

Proof. Given $\hat{\mathcal{L}}$, is a fuzzy congruence and by lemma 2.4, $S/\hat{\mathcal{L}}$, is an inverse semigroup.

Here
$\sigma_{\hat{\mathcal{L}}} = \{\hat{\mathcal{L}}_a, \hat{\mathcal{L}}_b) \in S/\hat{\mathcal{L}} \times S/\hat{\mathcal{L}} : \hat{\mathcal{L}}_{ea} = \hat{\mathcal{L}}_{eb}$ for some $\in E_s\}$.
By lemma 2.4, $\lambda_{\sigma\hat{\mathcal{L}}}$ is a group fuzzy congruence relation on $S/\hat{\mathcal{L}}$.

Theorem 3.9. *Let S be an inverse semigroup, $\hat{\mathcal{R}}$ is a fuzzy congruence Green's relation on S and $\mu_{\hat{\mathcal{R}}}$ a fuzzy congruences on $S/\hat{\mathcal{R}}$.*

Then the fuzzy relation $\lambda_{\sigma/\hat{\mathcal{R}}}$ is a group fuzzy congruence on $S/\hat{\mathcal{R}}$.

Proof. Given $\hat{\mathcal{R}}$ a fuzzy congruence. Then by lemma 2.4 $S/\hat{\mathcal{R}}$ is an inverse semigroup.
Here

$\sigma_{\hat{\mathcal{L}}} = \{\hat{\mathcal{L}}_a, \hat{\mathcal{L}}_b) \in S/\hat{\mathcal{L}} \times S/\hat{\mathcal{L}} : \hat{\mathcal{L}}_{ea} = \hat{\mathcal{L}}_{eb}$ for some $e \in E_s\}$.

Then, by lemma 3.4 $\lambda_{\sigma/\hat{\mathcal{R}}}$ is a group fuzzy congruence relation on $S/\hat{\mathcal{R}}$.

Theorem 3.10. *Let S be an inverse semigroup, $\hat{\mathcal{L}}$ is a fuzzy congruence Green's relation on S and γ a fuzzy congruence relation on $S/\hat{\mathcal{L}}$. Then γ is a group fuzzy congruence if and only if $\sigma_{\hat{\mathcal{L}}} \subseteq \gamma^{-1}(1)$, where $\sigma_{\hat{\mathcal{L}}}$ is the least group fuzzy congruence on $S/\hat{\mathcal{L}}$.*

Proof. Given $\hat{\mathcal{L}}$ is a fuzzy congruence. By lemma 2.4, $S/\hat{\mathcal{L}}$ is an inverse semigroup. Suppose γ is a group fuzzy congruence on $S/\hat{\mathcal{L}}$ $((=S^*)$

Then

$(\hat{\mathcal{L}}_a, \hat{\mathcal{L}}_b) \in \sigma_{\hat{\mathcal{L}}} \Rightarrow \hat{\mathcal{L}}_{ea} = \hat{\mathcal{L}}_{eb}$ some $e \in E_s$.

Since γ is a group fuzzy congruence on $S/\hat{\mathcal{L}}$ by definition 3.1, S^*/γ is a group. Here any element in S^* is of the form $\hat{\mathcal{L}}_x$ and any element

$\gamma\hat{\mathcal{L}}_x$. Since S^*/γ is a group, there exists on identity in S^*/γ is of the form $\gamma\hat{\mathcal{L}}_e$ is S^*/γ is that

$\gamma\hat{\mathcal{L}}_a = \gamma\hat{\mathcal{L}}_e * \gamma\hat{\mathcal{L}}_a = \gamma\hat{\mathcal{L}}_{ea} = \gamma\hat{\mathcal{L}}_{eb} = \gamma\hat{\mathcal{L}}_e * \gamma\hat{\mathcal{L}}_b = \gamma\hat{\mathcal{L}}_b$

That is,

$\gamma\hat{\mathcal{L}}_a = \gamma\hat{\mathcal{L}}_b$

Since γ is a fuzzy congruence

$\gamma\hat{\mathcal{L}}_a = \gamma\hat{\mathcal{L}}_b \Rightarrow \gamma(\hat{\mathcal{L}}_a, \hat{\mathcal{L}}_b) = 1 \Rightarrow (\hat{\mathcal{L}}_a, \hat{\mathcal{L}}_b) \in \gamma^{-1}(1)$

Hence

$(\hat{\mathcal{L}}_a, \hat{\mathcal{L}}_b) \in \sigma_L \Rightarrow (\hat{\mathcal{L}}_a, \hat{\mathcal{L}}_b) \in \gamma^{-1}(1)$.

So, $\sigma_L \leq \gamma^{-1}(1)$. Hence the result.

Conversely, assume $\sigma_L \leq \gamma^{-1}(1)$. Let $e, f \in E_s$. Then $(\hat{\mathcal{L}}_e, \hat{\mathcal{L}}_f) \in E_{S/\hat{\mathcal{L}}}$. Since $S/\hat{\mathcal{L}}$ is an inverse semigroup by lemma 3.3, $E_{S/\hat{\mathcal{L}}}$ is commutative and $\hat{\mathcal{L}}_{efe} \in E_{S/\hat{\mathcal{L}}}$ where $efe \in E_s$. Again, by Note 3.7.

$(efe)e = (efe)f$. Then we have $\hat{\mathcal{L}}(efe)e = \hat{\mathcal{L}}(efe)f$.

That is,

$\hat{\mathcal{L}}_{e'e} = \hat{\mathcal{L}}_{e'f}$ where $e' = efe \in E_S \Rightarrow (\hat{\mathcal{L}}_e, \hat{\mathcal{L}}_f) \in \sigma_{\hat{\mathcal{L}}}$

Since $\sigma_{\hat{\mathcal{L}}} \leq \gamma^{-1}(1)$. $(\hat{\mathcal{L}}_e, \hat{\mathcal{L}}_f) \in \gamma^{-1}(1)$. Therefore

$\gamma(\hat{\mathcal{L}}_e, \hat{\mathcal{L}}_f) = 1 \Rightarrow \gamma\hat{\mathcal{L}}_e = \gamma\hat{\mathcal{L}}_j$

Then by proposition 3.2, γ is a group fuzzy congruence on $S^* = S/\hat{\mathcal{L}}$.

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