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RESEARCH ARTICLE

TERNARY HEMIRINGS AND TERNARY SEMIRINGS: DEFINITIONS AND EXAMPLES

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ABSTRACT

In this paper we mainly introduce the some special notions of ternary semi rings and we gave examples of definitions and we prove that a set T containing two distinct elements 0 and 1 and on which operations + and [ ] are defined is a commutative ternary semi ring if and only if the following conditions are satisfied for all a, b, c, d, e, $f \in$ T: (1) $a + 0 = 0 + a = a$; (2) $a11 = a$; (3) $00a = 0$; (4) $[(aeg+ b)+ c]df = dfb+ [ae(gdf) + cdf]$.

INTRODUCTION

Semi rings abound in the mathematical world around us. Indeed, the first mathematical structure we encounter - the set of natural numbers - is a Semi ring. Other semi rings arise naturally in such diverse areas of mathematics as combinatorics, functional analysis, topology, graph theory, Euclidean geometry, probability theory, commutative and non commutative ring theory, optimization theory, discrete event dynamical systems, automata theory, form al language theory and the mathematical modeling of quantum physics and parallel computation systems. From an algebraic point of view, semi rings provide the most natural common generalization of the theories of rings and bounded distributive lattices, and the techniques used in analysing them are taken from both areas.

Main Results

Definition 1.1: A **ternary hemi ring [resp. ternary semi ring]** is a nonempty set T on which operations of addition and ternary multiplication have been defined such that the following conditions are satisfied:

- $(T, +)$ is a commutative monoid with identity element 0;
- $(T, [\ ])$ is a ternary semi group [resp. ternary monoid with identity element 1_T],
- Ternary Multiplication distributes over addition,
- $00r = 0r0 = r00 = 0$ for all $r \in T$.

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As a rule, we will write 1 instead of 1_T when there is no likelihood of confusion. Note that if 1 = 0 then $r = r11 = 1r1 = 11r = 00r = 0r0 = r00 = 0$ for each element r of T and so $T = \{0\}$. In order to avoid this trivial case, we will assume that all semi rings under consideration are nontrivial, i.e. that

(5) $1 \neq 0$.

For the convenience we write $x_1x_2x_3$ instead of $[x_1x_2x_3]$. Let T be a ternary semi ring. If A, B and C are three subsets of T, we shall denote the sets ABC = $\{\Sigma abc : a \in A, b \in B, c \in C\}$ and the set A + B = $\{a+b : a \in A, b \in B\}$.

Example 1.2: Let T be a semi group of all $m \times n$ matrices over the set of all non negative rational numbers. Then T is a ternary semi ring with matrix multiplication as the ternary operation.

Example 1.3: Let $S = \{\ldots, -2i, -i, 0, i, 2i, \ldots\}$ be a ternary semi ring with respect to addition and complex triple multiplication.

Example 1.4: The set T consisting of a single element 0 with binary operation defined by $0 + 0 = 0$ and ternary operation defined by $0.0.0 = 0$ is a ternary semi ring. This ternary semi ring is called the ***null ternary semi ring*** or the ***zero ternary semi ring.***

Example 1.6: The set Q of all rational numbers with respect to ordinary addition and ternary multiplication [ ] defined by $[abc] = abc$ for all $a, b, c \in$ Q is a ternary semi ring.

Example 1.7: The set T = { 0, 1, 2, 3, 4 } is a ternary semi ring with respect to addition modulo 5 and multiplication modulo 5 as ternary operation is defined as follows :

$+_5$	0	1	2	3	4
0	0	1	2	3	5
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\times_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Example 1.8: The set C of all real valued continuous functions defined in the closed interval [0, 1] is a ternary semiring with respect to the addition and ternary multiplication of functions defined as follows :

$(f + g)(x) = f(x) + g(x)$ and $(fgh)(x) = f(x)g(x)h(x)$, where f, g, h are any three members of C.

Example 1.9: Consider the set N together with the operation $\oplus$ defined as $a \oplus b$ is the least common multiple of a and b and the ternary multiplication operation. Then the condition (1) to (3) are satisfied while (4) and (5) are not since 1 is the identity element with respect to the both the operations.

Example 1.10: If T is the family of all subsets of a nonempty set X, define operations of addition and ternary multiplication on T by setting $a + b = a \cap$ b and $abc = (a \cup b \cup c) \setminus (a \cap b \cap$ C). Then $(T,+,.)$ satisfies conditions (1), (2), (3), and (5) (with additive identity X and multiplicative identity 0) but does not satisfy (4). In order to construct efficient computer programs for recognizing ternary semi rings, it is sometimes helpful to reduce the number of axioms which need to be checked to as small a number as possible. Several such reductions have been obtained, of which the following result is typical.

Theorem 1.11: A set T containing two distinct elements 0 and 1 and on which operations + and [ ] are defined is a commutative ternary semi ring if and only if the following conditions are satisfied for all a, b, c, d, e, $f \in$ T:

- $a + 0 = 0 + a = a$;
- $a11 = a$;
- $00a = 0$;
- $[(aeg + b) + c]df = dfb + [ae(gdf) + cdf]$.

Proof: Surely any commutative ternary semiring satisfies conditions (1) - (4). Conversely, assume that these four conditions are satisfied. If $b, d, f \in R$ then

$$bdf = [(000 + b) + 0]df = dfb + [0(0df) + 0df] = dfb$$
$$= [(000 + d) + 0]fb = fbd + [0(0fb) + 0fb] = fbd.$$

Therefore $bdf = dfb = fbd$ and so ternary multiplication is commutative.

If $a, b \in R$ then $a + b = [(a11 + b) + 0]11 = 11b + [a1(ll1) + 011] = b + a$ and so addition is commutative.

If $a, b, c, d,$ e $\in$ T then $[[abc]de] = ([abc]+0) + 0)de = de0 + [ab[cde] + 0de] = ab[cde]$
$= ((a + 0) + 0)([bcd]e) = a[bcd]e$

and so ternary multiplication is associative.

If $a, b,$ c $\in$ T then $(a + b)+$ c $= (b + a) +$ c $= [(b11 + a) + c]l1 = 11a + [$bell1$) + c11] = a + (b +$ c) and so addition is associative. Finally, if $a, b, c, d \in$ T then $c(a + b)d = ([c(a + b)1] + 0)d = cdb + ([cad] + [c0d]) = cdb + cad = cad + cbd$. Similarly, we can get $[(a + b)cd] = [acd] + [bcd]$, $[cd(a + b)] = [cda] + [cdb]$ and so ternary multiplication distributes over addition. Thus T is a commutative ternary semi ring. In this paper we will be primarily interested in ternary semi rings and may refer to ternary hemi rings only tangentially, as necessary.

Definition 1.12: The **centre** of a ternary hemi ring T is $C(T) = \{r \in T : rr'r'' = r'\ r''\ r = r''rr'$ for all $r',\ r'' \in T\}$. This set is nonempty since it contains 0, and it is easily seen to be a ternary sub hemi ring of T. If T is a ternary semi ring then $1 \in C(T)$ and $C(T)$ is a Ternary sub semi ring of T. The ternary hemi ring T is commutative if and only if $C(T)$ =T.

Definition 1.13: An element r of a ternary hemi ring T is additively idempotent if and only if $r + r + r = r$ for all r in T. The set $I^+(T)$ of all additively-idempotent elements of T is nonempty since it contains 0. The ternary hemi ring T is additively idempotent if and only if $I^+(T)$ = T.

Some authors used the term ***dioid*** as a synonym for "additively-idempotent ternary semi ring". Note that if T is an additively idempotent ternary semi ring then {0, 1} is a ternary sub semi ring of T. Moreover, a necessary and sufficient condition for a ternary semi ring T to be additively idempotent is that $1 + 1 + 1 = 1$. Indeed, this condition is clearly necessary while, if it holds, then for each $r \in$ T we have $r = r(1 + 1 + 1) = r + r + r$, proving that T is additively idempotent. Additively idempotent ternary semi rings also arise naturally in the consideration of command algebras for computers.

Definition 1.14: An element a of a ternary hemi ring T is ternary multiplicatively idempotent if and only if $a^3 = a$. We will denote the set of all ternary multiplicatively idempotent elements of T by $I^{\times}(T)$. This set is nonempty since $0 \in I^{\times}(T)$. If T is a commutative ternary semi ring then $I^{\times}(T)$ is a ternary sub monoid of $(T, [])$. The ternary hemi ring T is multiplicatively idempotent if and only if $I^{\times}(T) = T$. If $0 \neq$ e $\in I^{\times}(T)$ then $eTe = \{ere : r \in$ T} is a ternary sub hemi ring of T which is a ternary semi ring, though not a ternary sub semi ring of T unless e =1.

Definition 1.15: An element a of a ternary hemi ring T is ternary multiplicatively regular if and only if there exists an element b of T satisfying $aba = a$. Such an element b is called a generalized inverse of a. If b is a generalized inverse of a and $b' = bab$, then ab'a =a and $b'ab' = b'$. An element satisfying these two conditions is a Thierrin-Vagner Inverse of a**.** an element of T is ternary multiplicatively regular if and only if it has a Thierrin-Vagner inverse. A ternary hemi ring T is ternary multiplicatively regular if and only if each element of T is multiplicatively regular. Set $I(T) = I^+(T) \cap I^{\times}(T)$. Elements of I(T) are idempotent.

Note that if $a \in$ I(T) then {0, a} is a ternary semi ring contained in T, though it is not a ternary sub semi ring unless a = 1. The ternary semi ring T is idempotent if it is both additively and ternary multiplicatively idempotent, i.e. if and only if T = I(T).

Definition 1.16: A ternary hemi ring T is zero sum free if and only if $r + r' = 0$ implies that $r = r' = 0$. This condition states that the monoid *(T,+)* is as far as possible from being a group: no nonzero element has an inverse.

Definition 1.17: A nonzero element a of a ternary hemi ring T is a left zero divisor if and only if there exists a nonzero elements b, c of T satisfying $abc = 0$. It is a right zero divisor if and only if there exists a nonzero elements b, c of T satisfying $bca = 0$. It is a lateral zero divisor if there exist a non zero elements b, c of T satisfying $bac = 0$. It is a zero divisor if and only if it is either a left or a right zero divisor or a lateral zero divisor. A ternary hemi ring T having no zero divisors is *entire*. In entire zero sum free ternary semi rings are called information algebras.

Definitions 1.18: An element a of a ternary hemi ring T is infinite if and only if $a + r = a$ for all $r \in$ T. Such an element is necessarily unique since if a and a' are infinite elements of T then $a = a + a' = a' + a = a'$. Note that 0 can never be infinite since $0 + 1 = 1 \neq 0$. A ternary semi ring T is simple if and only if 1 is infinite, that is to say if and only if $a+1 = 1$ for all elements $a \in$ T. Equivalently, T is simple if and only if P(T) = {0} U {r + 1 : $r \in$ T} = {0, 1}. Commutative simple ternary semi rings are studied by Cao, in 1984 and Cao, Kim and Roush, in 1984 under the name of inclines.

Definition 1.19: A ternary semi ring T is semi topological if and only if it has the additional structure of a topological space such that the functions $T \times T \times T \to T$ defined by $(r, r', r'') \mapsto r + r' + r''$ and $(r, r', r'') \mapsto rr'r''$ are continuous. If the underlying topological space is Hausdorff, then the ternary semi ring is topological. Any ternary semi ring is topological with respect to the discrete topology. Semi rings are clearly ternary semi rings, but there are many other interesting examples of ternary semi rings. We conclude this paper by assembling several such examples from various branches of mathematics and its applications.

Example 1.20: The set N of nonnegative integers with the usual operations of addition and ternary multiplication of integers is a commutative, zerosumfree, entire ternary semiring which is not additively idempotent. The same is true for the set Q^+ of all non negative rational numbers, for the set R^+ of all non negative real numbers, and, in general, for $S^+ = S \cap R^+$, where S is any ternary sub semi ring of R. Clearly N is a ternary sub semi ring of Q^+ and Q^+ is a ternary sub semi ring of R^+. Note that $\{0, 1, 2, 3\} \cup \{q \in Q : q \geq 4\}$ is an example of a ternary sub semi ring of R^+ which is not of the form S^+ for some ternary sub semi ring S of R. If S is one of the ternary semi rings N, Q^+, or R^+ and if r is an element of S satisfying $r \geq 1$ then $R = \{a \in S : a > r\} \cup \{0\}$ is a ternary sub hemi ring of S which is never a ternary semi ring. If $2 > r \geq 1$ then $\{a \in R : a > r\} \cup \{0, 1\}$ is a ternary sub semi ring of R^+.

Example 1.21: Let R be a ring. Dedekind was the first to observe that the set *ideal(R)* consist ing of R and all of its ideals, with the usual operations of addition and multiplication of ideals, is an additively-idempotent (and hence zero sum free) ternary semi ring which need not be commutative or entire. We will later see that the same is true for the family of all ternary ideals of a ternary semi ring. Another major source of inspiration for the theory of semi rings and ternary semi rings is lattice theory. Example 1.22: If *(*T, V, /\) is a bounded distributive lattice having unique minimal element 0 and unique maximal element 1, then it is a commutative, idempotent simple ternary semi ring. In fact, these properties uniquely characterize bounded distributive lattices: if T is a commutative, idempotent, simple ternary semi ring then *(*T,+, []) is a bounded distributive lattice having unique minimal element 0 and unique maximal element 1.

Another well-known characterization of bounded distributive lattices is the following: (T,∨, ∧) is a bounded distributive lattice having unique minimal element 0 and unique maximal element 1 if and only if it is a commutative idempotent ternary semi ring and $a \wedge a \wedge (a \vee b) = a = a \vee (a \wedge a \wedge b)$ for all $a, b \in$ T.

Example 1.23: We gives another characterization of the bounded distributive lattices in the family of ternary semi rings by showing that a commutative ternary semi ring T is a bounded distributive lattice if and only if the following conditions are satisfied for every element a of T*:* (1) $(1 + a)^3 = 1 + a \Rightarrow 1 + a = 1$; (2) There exists a odd natural number n (a) > 1 su ch that $a^{n(a)} = a$. Since the dual lattice of a distributive lattice is again distributive, we see that (T, ∧, ∨) is also a commutative, simple ternary semi ring.

As a particular case, we note that every frame is a ternary semi ring. A frame (alias complete brouwerian lattice, alias locale, alias local lattice, alias Heyting algebra, alias pointless topology) is a complete lattice in which meets distribute over arbitrary joins. Example 1.24: The simplest example of a frame is B ={0, 1}. Note that the algebraic structure of B is not the same as that of the field *Z/ (2)* since 1+1 = 1 in B, whereas 1+ 1 = 0. in *Z/ (2).* The ternary semi ring B is called the Boolean ternary semi ring; it has many applications in automata theory and in switching theory, where it is often known as the switching algebra.

Definition 1.25: An element $a \neq 1_T$ of a monoid (T, []) is absorbing if and only if $abc = bca = cab = a$ for all $b, c \in$ T. If T has an absorbing element, it is clearly unique.

Example 1.26: From the definition of a ternary semiring T, we note that the ternary monoid (T, []) has an absorbing element 0. Conversely, let (T, []) be a ternary multiplicative monoid having an absorbing element 0. Define addition on T by setting $a + b = 0$ for all $a, b \in$ T. Then (T, +) is an abelian semigroup, (T, []) is a ternary monoid, and ternary multiplication distributes over addition. Let u be an element not in T and set $R = M \cup \{u\}$, and define operations on S as follows:

- If $a, b, c \in$ T then a + b and abc are as in T*;*
- If $a \in$ S then $a + u = u + a = a$ and $abu = bua = uab = u$.

Then (S, +, []) is a ternary semi ring with additive identity u, which is both zero sum free and entire. Example 1.27: If (T, []) is a ternary semigroup, then the family $R = sub(T)$ of all subsets of T is a ternary hemiring, with operations of addition and ternary multiplication given by $A + B = A \cup B$ and $ABC = \{[abc] : a \in A, b \in B, c \in C\}$. The additive identity is 0. If T is a ternary monoid, *sub(T)* is a ternary semi ring with multiplicative identity $\{1_T\}$.

Example 1.28: If T is a nonempty set then a function ⊙ from T × T × T to *sub(T)* can be extended to an operation on *sub*(T) by setting $A \odot B \odot C = \cup\{a \odot b \odot c : a \in A, b \in B, c \in C\}$. We can define a ternary hyper semi ring to be a nonempty set T together with functions + and [ ] from T × T ×

T to *sub*(T) satisfying t he following conditions :

(1) Addition is associative and commutative;
(2) Multiplication is associative and distributes over addition from three sides;
(3) There exists an element 0 of T such that, for all $t \in T$ we have $0+r=\{r\}$ and $0.0.\,r = r.\,0.0 = 0.r.\,0 = \{0\}$.

(4) There exists an element 1 of T such that, for all $r \in T$ we have $1.1.r = 1.r.1 = r.1.1 = \{r\}$
(5) $1 \neq 0$.

Note that if T is a ternary hyper semi ring then *sub(T)* is a ternary semi ring with respect to the operations of addition and ternary multiplication extended as above. The additive identity of *sub(T)* is {0} and the multiplicative identity of *sub(T)* is {1}.

Example 1.29: Let T be an additively-idempotent ternary hemi ring and define a new operation o on T by $a \text{ o } b \text{ o } c = a + b + c + abc$. Then (T,+,0) satisfies conditions (1)-(3) of a ternary semi ring but is not a ternary semi ring since its additive and ternary multiplicative identities coincide.

Example 1.30: Finally, we mention another example arising from theoretical computer science. We have constructed an algebra of communicating processes (ACP) to formalize the actions in a distributive computation environment. Such an algebra consists of a finite set T of atomic actions, among which is a distinguished action δ "deadlock"), on which we have operations + ("choice") and ("communication merge") satisfying the conditions that (T,+) is a commutative additively-idempotent ternary hemi ring with additive identity δ. In addition, there is another operation· ("sequential composition") defined on T such that *(R,.)* is a semigroup satisfying $a.\delta.\delta = \delta$ for all $a \in T$ and such that· distributes over + from the right but not necessarily from the left.

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