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RESEARCH ARTICLE

ON THE 4-CLIFFORD ALGEBRA IN ABSTRACT DIFFERENTIAL GEOMETRY

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ABSTRACT

In this research paper, differential triads over a topological space X are defined as objects of the Cartesian product category $Alg \times_X F^d \times_X DMod$ over the same fixed topological space X. From differential triads sheaves $(\mathcal{A}_{iX}, \partial_{iX}, \Omega_{iX})$ and $(\mathcal{A}_{jX}, \partial_{jX}, \Omega_{jX})$ we defined morphisms of differential triads using a differential morphism $\partial^{ij}: H^{ij}_{\mathcal{A}} \to H^{ij}_{\Omega}$, where $H^{ij}_{\mathcal{A}} = Hom_{Alg_X}(\mathcal{A}_{iX}, \mathcal{A}_{jX})$ and $H^{ij}_{\Omega} = Hom_{DMod_X}(\Omega_i, \Omega_j)$. From the quadratic forms, we determined quadratic differential triads and Clifford differential triads.

1. INTRODUCTION

We consider X as a topological space (see[4] &[5]), Ω_X as a sheaf of (differential) $\mathcal{A}_X$-modules over X, ∂_X as a derivative map like the $\mathbb{K}_X$ −sheaf morphism which is also $\mathbb{K}_X$ −linear, where $\mathbb{K} = (\mathbb{R} \equiv (\mathbb{R}, r, X)\ and\ \mathbb{C} \equiv (\mathbb{C}, c, X))$ is respectively the sheaf of real numbers and the sheaf of complex numbers and $\mathcal{A}_X$ the sheaf of unital $\mathbb{K}$ −algebras over X (see[10], [11], [12] &[13]). The triplet

$$(\mathcal{A}_X, \partial_X, \Omega_X) \tag{1.1}$$

which satisfies, for any open U in X, the Leibniz (product) rule [6]

$$\partial_U(a.w) = a.\partial_U(w) + w.\partial_U(a) \tag{1.2}$$

with $a, w \in \mathcal{A}_U$, and $d_U: \mathcal{A}_U \to \Omega_U$ is continuous. We set

$$dT_X = (\mathcal{A}_X, \partial_X, \Omega_X) \tag{1.3}$$

and say that dT_X is a differential triad relative to $(X, \mathcal{A}_X)$. If $dT_{iX} = (\mathcal{A}_{iX}, \partial_{iX}, \Omega_{iX})$ and $dT_{jX} = (\mathcal{A}_{jX}, \partial_{jX}, \Omega_{jX})$ are two differential triads respectively, relative to $(X, \mathcal{A}_{iX})$ and $(X, \mathcal{A}_{jX})$, then a morphism of differential triads between dT_{iX} and dT_{jX} (or simply from dT_{iX} to dT_{jX}) is the following triplet

$$(h^{ij}_{\mathcal{A}_X}, \partial^{ij}_X, h^{ij}_{\Omega_X}) \tag{1.4}$$

where $h^{ij}_{\mathcal{A}_X} \in Hom_{Alg_X}(\mathcal{A}_{iX}, \mathcal{A}_{jX})$ and $h^{ij}_{\Omega_X} \in Hom_{DMod_X}(\Omega_{iX}, \Omega_{jX})$ are continuous maps and ∂^{ij}_X is such that for any open U in X we have

$$\partial^{ij}_U(h^{ij}_{\mathcal{A}_U}) = h^{ij}_{\Omega_U} \tag{1.5}$$

We design by $\mathcal{A}lg_X$ the category of sheaves of unital $\mathbb{K}_X$-algebras over X and $DMod_X$ the category of sheaves of (differential) modules over X, where $\mathbb{K}_X = (\mathbb{R}_X\ or\ \mathbb{C}_X)$, with $\mathbb{R}_X = (\mathbb{R}, r, X)$ the sheaf of real numbers and $\mathbb{C}_X = (\mathbb{C}, c, X)$ the sheaf of complex numbers over X (see[8]&[9]).

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The set of differentials over $X \in TOP$ is represented by $Diff_X$, it is also considered as a category of differential morphisms. Let us consider the triplet (see[25])

$$(Alg_X, Diff_X, DMod_X) \tag{1.6}$$

such that, for any $\mathcal{A}_{iX} \in Ob(Alg_X)$, there exist $\partial_{iX} \in Diff_X$ and $\Omega_{iX} \in Ob(DMod_X)$ satisfying the Leibniz (product) rule

$$\partial_{iU}(a_i.a'_i) = a_i.\partial_{iU}(a'_i) + a'_i.\partial_{iU}(a_i) \tag{1.7}$$

with $a_i, a'_i \in \mathcal{A}_{iU} \equiv \mathcal{A}_i(U)$, where $\partial_{iU}: \mathcal{A}_{iU} \to \Omega_{iU} \equiv \Omega_i(U)$ is continuous and $\mathbb{K}_U$-linear. The differential triad dT_X over $(X, \mathcal{A}_X)$ is given by

$$dT_X = (\mathcal{A}_X, \partial_X, \Omega_X) \tag{1.8}$$

The application $F_X^d: Alg_X \to DMod_X$ is a functor defined, for any $\mathcal{A}_{iX}, \mathcal{A}_{jX} \in Ob(Alg_X)$ and $h_{\mathcal{A}_X}^{ij} \in H_{\mathcal{A}_X}^{ij}$ as follows

$$F_x^d|_{\mathcal{A}_{iX}} = \partial_{iX} \text{ and } F_x^d|_{H_{\mathcal{A}_X}^{ij}} = \partial_X^{ij} \tag{1.9}$$

where $H_{\mathcal{A}_X}^{ij} = Hom_{Alg_X}(\mathcal{A}_{iX}, \mathcal{A}_{jX})$ and $\partial_X^{ij}: H_{\mathcal{A}_X}^{ij} \to H_{\Omega_X}^{ij}$ is a continuous map , with $H_{\Omega_X}^{ij} = Hom_{DMod_X}(\Omega_{iX}, \Omega_{jX})$. The symbol "$|$" designs the restriction, and we say in this case that the triplets

$$(\mathcal{A}_{iX}, \partial_{iX}, \Omega_{iX}) \text{ and } \left(H_{\mathcal{A}_X}^{ij}, \partial_X^{ij}, H_{\Omega_X}^{ij}\right) \tag{1.10}$$

are respectively differential triads in $Ob(Alg \times_X F^d \times_X DMod)$ and $Mor(Alg \times_X F^d \times_X DMod)$. The functor given by $F_X^d: Alg_X \to DMod_X$ is a differential triad functor over *X*. We have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A}_{iU} & \xrightarrow{\partial_{iU}} & \Omega_{iU} \\ \varinjlim_{x \epsilon U} \downarrow & & \downarrow \varinjlim_{x \epsilon U} \\ \mathcal{A}_{ix} & \xrightarrow[\partial_{ix}]{} & \Omega_{iX} \end{array}$$

For this regard, we have the following inductive limit

$$\varinjlim_{x \epsilon U} \ o \ \partial_{iU} = \varinjlim_{x \epsilon U} \ o \ \partial_{ix} \tag{1.11}$$

The differential triads dT_{iX} and their morphisms mdT_X^{ij} with *i,j = 1,2,3,...* represent the category which is denoted by $DiffT_X$ and called the category of differential triads over *X* (see[14]).

The same notions can be generalized over categories $Open_X$ and TOP and we attend to construct the category of differential triads over $Open_X$ and TOP, respectively denoted, by

$$DiffT_{Open_X} \text{ and } DiffT_{TOP} = DiffT$$

with

$$DiffT_X \subseteq DiffT_{Open_X} \subseteq DiffT.$$

A complex of free left (resp. right) $\mathcal{A}$-modules is a sequence of left (resp. right) the following $\mathcal{A}$-homomorphisms (see[6])

$$\Omega^* \equiv \cdots \xrightarrow{d^{i-1}} \Omega^i \xrightarrow{d^i} \Omega^{i+1} \xrightarrow{d^{i+1}} \ldots \tag{1.12}$$

between left (resp. right) $\mathcal{A}$-modules Ω^i and Ω^{i+1} which satisfy, for any open *U* in *X*,

$$Im\, d^{i-1}(U) \subseteq Ker\, d^i(U), i.e., d_U^i o\, d_U^{i-1} = 0_U \quad \forall i \in \mathbb{Z} \tag{1.13}$$

Regarding the above composition (1.13), where by definition

$$d^0 = \partial, d^i = d \quad \text{for any } i \geq 1 \tag{1.14}$$

where the symbol d designs the differential $\mathcal{A}$-homomorphism**.**

If we specify the order of the set Ω of differential forms (sheaf of differential $\mathcal{A}$-modules) by setting

$$\Omega^0 = \mathcal{A} \text{ and } \Omega^i \equiv (\Omega^1)^i = \wedge^i \Omega^1 \tag{1.15}$$

where $\wedge \equiv \wedge_{\mathcal{A}}$ is the exterior (the skew symmetric homological tensor) product, for any $i \geq 1$. We have, more explicitly

$$\begin{cases} \Omega^1 = \mathcal{A} \wedge \Omega \\ \Omega^2 = \mathcal{A} \wedge \Omega^1 \wedge \Omega^1 \end{cases} \tag{1.16}$$

2. The quadratic differential triads

Let dT_X be a differential triad. If $(\mathcal{A}_X, q_{\mathcal{A}_X})$ and $(\Omega_X\ q_{\Omega_X})$ are two $\mathcal{A}_X$ -quadratic spaces, with $q_{\mathcal{A}_X} : \mathcal{A}_X \to \mathcal{A}_X$ and $q_{\Omega_X} : \Omega_X \to \Omega_X$ two quadratic forms, then the triplet $((\mathcal{A}_X, q_{\mathcal{A}_X}), \partial_X, (\Omega_X\ q_{\Omega_X}))$ is an $\mathcal{A}_X$-quadratic differential triad *iff* the following relation is verified

$$q_{\Omega_U}\ o\ \partial_U = q_{\mathcal{A}_U} \tag{2.1}$$

According to the above expression, it follows that $q_{\mathcal{A}_X}$ represents a differential quadratic form. The quadratic differential triad relative to $(X, \mathcal{A}_X)$ is determined as

$$qdT_X \equiv ((\mathcal{A}_X, q_{\mathcal{A}_X}), \partial_X, (\Omega_X, q_{\Omega_X})) \tag{2.2}$$

Here qdT_{iX} and qdT_{jX} are two quadratic differential triads relative to $(X, \mathcal{A}_{iX})$ and $(X, \mathcal{A}_{jX})$, respectively , with *i, j = 1 ,2, …*A morphism of quadratic differential triads from qdT_{iX} into qdT_{jX} is the triplet

$$(h^{ij}_{\mathcal{A}_X}, \partial^{ij}_X, h^{ij}_{\Omega_X}) \tag{2.3}$$

where $h^{ij}_{\mathcal{A}_X} \in H^{ij}_{\mathcal{A}_X} = Hom_{\mathbb{K}_X}(\mathcal{A}_{iX}, \mathcal{A}_{jX})$ and $h^{ij}_{\Omega_X} \in H^{ij}_{\Omega_X} = Hom_{\mathcal{A}_X}(\Omega_{iX}, \Omega_{jX})$ and moreover the following relation

$$\partial^{ij}_X(h^{ij}_{\mathcal{A}_X}) = h^{ij}_{\Omega_X} \tag{2.4}$$

satisfies the Leibniz (product) rule, as given in (1.7). We observe that the triplet $(H^{ij}_{\mathcal{A}_X}, \partial^{ij}_X, H^{ij}_{\Omega_X})$ represents the differential triad. The categories of differential triads and quadratic differential triads over X are denoted respectively by

$$DiffT_X,\ QDiffT_X \tag{2.5}$$

(D_1) In the context of sheaves over categories, we intend to replace the topological space X, respectively, by the categories $Open_X$ and *TOP* so that we determine, respectively, the categories of quadratic differential triads over $Open_X$ and *TOP*, denoted by

$$QDiffT_{Open_X},\quad QDiffT_{TOP} \equiv QDiffT \tag{2.6}$$

(D_2) Is it possible to express a functor $Q: DiffT \to QDiffT$ as follows

$$Q(\mathcal{A}, \partial, \Omega) = ((\mathcal{A}, q_{\mathcal{A}}), \partial, (\Omega, q_\Omega)) \text{ and } Q(h^{ij}_{\mathcal{A}}, \partial^{ij}, h^{ij}_\Omega) = (h^{ij}_{\mathcal{A}}, \partial^{ij}, h^{ij}_\Omega) \tag{2.7}$$

where $q_{\mathcal{A}}$ and q_Ω satisfy the composition condition

$$q_\Omega\ o\ \partial = q_{\mathcal{A}}\ .$$

(D_3) Our main concern is to find out what kind of pairs $(q_{\mathcal{A}}, q_\Omega)$ that can satisfy the above expression $q_\Omega\ o\ \partial = q_{\mathcal{A}}$?
Knowing that from a given pair $(\mathcal{A}, \Omega)$, we can define several pairs of quadratic spaces , i.e., $(\mathcal{A}, q^i_{\mathcal{A}})$, (Ω, q^i_Ω) , where *i = 1,2,3,…*

To answer the above concern, we need to fix a (differential) $\mathcal{A}$ -quadratic form $\dot{q}_\Omega: \Omega \to \mathcal{A}$ such that

$$\int \dot{q}_{\Omega} = \hat{q}_{\mathcal{A}} \tag{2.8}$$

where the symbol $\int \dot{q}_{\Omega}$ designs the "integral" of the differential form $\dot{q_{\Omega}}$, with $\hat{q}_{\mathcal{A}}$ the primitive function of $\dot{q_{\Omega}}$. In other terms, the primitive function is

$$\hat{q}_{\mathcal{A}} = q_{\mathcal{A}} + k \quad , \quad k \in \mathcal{A}_{\mathbb{R}} \tag{2.9}$$

(D_4) Designing by $\mathcal{A}_{\mathbb{K}}$ the underlying of $\mathbb{K}$ in $\mathcal{A}$ and then obtain an equivalence relation, ~, defined in $End_{\mathbb{K}}(\mathcal{A}) \equiv Hom_{\mathbb{K}}(\mathcal{A},\mathcal{A})$ as follows

$$q^i_{\mathcal{A}} \sim q^j_{\mathcal{A}} \quad \Leftrightarrow \quad q^i_{\mathcal{A}} - q^j_{\mathcal{A}} = k \quad \text{with} \quad k \in \mathcal{A}_{\mathbb{K}} \tag{2.10}$$

(D_5) Referring to (D_3), from (differential) $\mathcal{A}$ -quadratic form $\dot{q}_{\Omega}: \Omega \rightarrow \mathcal{A}$, we determine a subcategory $\dot{Q}DiffT$ of $QDiffT$ whose objects $\big((\mathcal{A}, \hat{q}_{\mathcal{A}}), d, (\Omega, \dot{q}_{\Omega})\big)$ verify at the same time, expressions (2.8), (2.9) and the following expression

$$\dot{q}_{\Omega} \, o \, \partial = \hat{q}_{\mathcal{A}} \tag{2.11}$$

Let us design by $\hat{Q}_{\mathcal{A}}$ and $\dot{Q}_{\Omega}$ the set of differential quadratic forms $\dot{q}_{\Omega}: \Omega \rightarrow \mathcal{A}$,… and the set of quadratic forms $\hat{q}_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}$, …, respectively. By setting

$$\int : \dot{Q}_{\Omega} \rightarrow \dot{Q}_{\mathcal{A}} \quad , \quad \dot{q}_{\Omega} \rightarrow \int \dot{q}_{\Omega} = \hat{q}_{\mathcal{A}} \tag{2.12}$$

Then the triplet

$$qinT = \big(\dot{Q}_{\Omega}, \int, \hat{Q}_{\mathcal{A}}\big) \tag{2.13}$$

represents the quadratic integral triad over $\mathcal{A}$ if and only if the following inductive limit is verified

$$\int_x = \underset{x \epsilon U}{\underline{lim}} \int_U = \underset{x \epsilon U}{\underline{lim}} \, (\partial_U)^{-1} = (\partial_x)^{-1} \tag{2.14}$$

where ∂ satisfies the Leibniz product rule. According to the terminology of (1.6), we can write expression (2.13) as

$$QINT_X = (QDMod_X, INT_X, QAlg_X) \tag{2.15}$$

where INT_X represents the set (or the category) of integral triads, $QINT_X$ is the category of quadratic differential modules and $QAlg_X$ is the category of algebras, all over *X*.

Considering the linear mapping $\dot{Q}: DiffT \rightarrow \dot{Q}DiffT \subseteq QDiffT$ which clearly defines an operator satisfying the expressions (2.8), (2.9) and (2.12), we observe that $\dot{Q}$ is a quadratic functorial operator.

Theorem 2.1 The quadratic functorial operator $\dot{Q}$ is a covariant functor.

Proof. Let $dT_i \,, dT_j, dT_k \in Ob(DiffT)$ and $mdT_{ij} \in Hom\big(dT_i, dT_j\big), mdT_{ik} \in Hom(dT_i, dT_k)$ and $mdT_{jk} \in Hom\big(dT_j, dT_k\big)$. Let us apply the operator $\dot{Q}$ on $DiffT$, so that

(1) $\dot{Q}(mdT_{jk} \circ mdT_{ij}) = m\dot{Q}dT_{jk} \circ m\dot{Q}dT_{ij} = \dot{Q}(mdT_{jk}) \circ \dot{Q}(mdT_{ij})$
(2) $\dot{Q}(id_{dT_i}) = id_{\dot{Q}dT_i} = id_{\dot{Q}(dT_i)}$

By convenience, we set the quadratic functorial operator as

$$\dot{Q}DiffT =< (DiffT \,, \dot{q} = \{\hat{q}_{\mathcal{A}} \,, \dot{q}_{\Omega}\}) > \tag{2.16}$$

where $\{\dot{q}_{\mathcal{A}} \,, \dot{q}_{\Omega}\}$ satidfies (2.1).

Analogously, we can construct the quadratic functor operators as

$$\dot{Q}_{Open_X} : DiffT_{Open_X} \rightarrow \dot{Q}DiffT_{Open_X} \tag{2.17}$$

and

$$\dot{Q}_{TOP} : DiffT_{TOP} \rightarrow \dot{Q}DiffT_{TOP} \tag{2.18}$$

For $H_{\mathcal{A}}^{ij} = Hom_{\mathbb{K}}(\mathcal{A}_i, \mathcal{A}_j)$ and $H_{\Omega^1}^{ij} = Hom_{\mathcal{A}}(\Omega_i^1, \Omega_j^1)$, the triplet $(H_{\mathcal{A}}^{ij}, \partial^{ij}, H_{\Omega}^{ij})$ is a differential triad over X, $Open_X$ or TOP then consequently the quadratic functorial operator $\dot{Q}$ acts on $(H_{\mathcal{A}}^{ij}, d^{ij}, H_{\Omega}^{ij})$ so that the quadratic differential triad becomes

$$\dot{Q}(dT^{ij}) = \dot{q}dT^{ij} = \left(\left(H_{\mathcal{A}}^{ij}, \dot{q}_{H_{\mathcal{A}}^{ij}}\right), \partial^{ij}, \left(H_{\Omega^1}^{ij}, \dot{q}_{H_{\Omega^1}^{ij}}\right)\right) \quad (2.19)$$

3. The main result

Our purpose now is to study the methods under which we associate the quadratic differential triad over $(X, (\mathcal{A}_X, q_{\mathcal{A}_X}))$ denoted by $qdT_X = ((\mathcal{A}_X, q_{\mathcal{A}_X}), \partial_X, (\Omega_X, q_{\Omega_X}))$ to a Clifford differential triad over $(X, (\mathcal{A}_X, q_{\mathcal{A}_X}))$ denoted by $CdT_X = (C(\mathcal{A}_X, q_{\mathcal{A}_X}), \partial_X^C, C(\Omega_X, q_{\Omega_X}))$. Designing the categories of Clifford differential triads over X, $Open_X$ and TOP by $CDiffT_X$, $CDiffT_{Open_X}$, and $CDiffT_{TOP}$ and letting (E_X, q_{E_X}) be a quadratic space, the Clifford algebra of (E_X, q_{E_X}) or simply, of E, is as a pair (C_X, c_X) formed by an $\mathcal{A}_X$–algebra C_X and an $\mathcal{A}$-linear map $c_X: E_X \rightarrow C_X$ such that , for any open $U \subseteq X$, we have (see[1], [2] ,[3] &[7])

$$c_U(s)^2 = q_{E_U}(s).1_{C_U} \quad (3.1)$$

where $s \in E_U \equiv E(U)$ and 1_{C_U} designs the unity in $C_U \equiv C(U)$. Also, for any $\mathcal{A}_X$-algebra F_X and all $\mathcal{A}_X$–linear map $f_X: E_X \rightarrow F_X$ such that, for any open $U \subseteq X$, we have

$$f_U(s)^2 = q_{E_U}(s).1_{F_U} \quad (3.2)$$

where $s \in E_U$, there exists a unique morphism $\sigma_X: C_X \rightarrow F_X$ of $\mathcal{A}_X$–algebras verifying $\sigma_U o c_U = f_U$. The Clifford $\mathcal{A}_X$–algebra (C_X, c_X) is denoted by

$$C_X \equiv C_X(E_X, q_{E_X}) \equiv (C_X(E_X, q_{E_X}), c_X) \quad (3.3)$$

Analogously, we construct Clifford $\mathcal{A}_X$–algebra through another approach, designing by $I(q_{E_X})$ the $\mathcal{A}_X$–ideal of the tensor $\mathcal{A}_X$–algebra $T(E_X)$ generated , for any open $U \subseteq X$, by the elements of the form

$$(s \otimes s).q_{E_U}(s).1_{T(E_U)}, \ s \in E_U \quad (3.4)$$

We restrict the graduation of $T(E_X)$ on Z/(2), the $\mathcal{A}_X$–ideal $I(q_{E_X})$ is homogeneous and the quotient

$$T(E_X)/I(q_{E_X}) \quad (3.5)$$

is a graded $\mathcal{A}_X$ –algebra on Z/(2) , such that the homogeneous elements of degrees o and 1 are easy to describe.
We design by

$$T_0(E_X) \equiv T_0(E_{0X}) \text{ and } T_1(E_X) \equiv T_1(E_{1X}) \quad (3.6)$$

respectively, the sub $\mathcal{A}_X$–algebra of homogeneous elements of degree 0 and the sub $\mathcal{A}_X$ –module (or sub-vector sheaf) of homogenous elements of degree 1. We set

$$c_X(E_X, q_{E_X})T(E_X)/I(q_{E_X}) \quad (3.7)$$

and the canonical projection $\pi_X: T(E_X) \rightarrow c_X(E_X, q_{E_X})$ is such that , for any open $U \subseteq X$, we have

$$(\pi_U(s))^2 = q_{E_U}(s).1_{T(E_U)} \quad (3.8)$$

with $s \in E_U$. As $T(E_X) = T_0(E_X) + T_1(E_X)$, we observe that

$$c_X(E_X, q_{E_X}) = c_{0X}(E_{0X}, q_{E_{0X}}) + c_{1X}(E_{1X}, q_{E_{1X}}) \quad (3.9)$$

where $c_{0X}(E_{0X}, q_{E_{0X}}) = \pi_X(T_0(E_{0X}))$ and $c_{1X}(E_{1X}, q_{E_{1X}}) = \pi_X(T_1(E_{1X}))$. It follows that $c_X(E_X, q_{E_X})$ is a Z/(2) –graded $\mathcal{A}_X$-algebra.

Theorem 3.1 Let $\delta_X: E_X \rightarrow D_X$ be an $\mathcal{A}_X$–linear map such that , for any open $U \subseteq X$, we have

$$(\delta_U(s))^2 = q_{E_U}(s).1_{C_U}, \ s \in E_U \quad (3.10)$$

Then, there exists a unique $\mathcal{A}_X$–algebra morphism $\varphi_X: C_X \to D_X$ such that

$$\left(\delta_U(s)\right)^2 = \varphi_U\left(\pi_U(s)\right) \tag{3.11}$$

Proof. By definition of tensor $\mathcal{A}_X$–algebra, there exists a unique map

$$\bar{\delta}_X: T(E_X) \to D_X$$

which extends δ_X , then , for $s \in E_U$, we have (see[23] & [24])

$$\begin{aligned}\bar{\delta}_U\left(s \otimes s - q_{E_U}(s).1_{T(E_U)}\right) &= \bar{\delta}_U(s \otimes s) - q_{E_U}(s).1_{T(E_U)} \\ &= \bar{\delta}_U(s)\bar{\delta}_U(s) - q_{E_U}(s).1_{T(E_U)} \\ &= \delta_U(s)\delta_U(s) - q_{E_U}(s).\varphi_U\left(\pi_U\left(1_{T(E_U)}\right)\right) \\ &= \left(\delta_U(s)\right)^2 - q_{E_U}(s).1_{D_U}\end{aligned}$$

It follows that $\delta_X\left(I(q_X)\right) = 0$ and $I(q_X) \subset ker(E_X)$. For these reasons, there exists a unique $\varphi_X: C_X \to D_X$ such that

$$\varphi_U \circ \pi_U = \bar{\delta}_U \text{ and } \bar{\delta}_U \circ t_U = \delta_U$$

where $\delta_U(s)$ is the contraction of $\bar{\delta}_U(s)$, for any $s \in E_U$, with $U \subseteq X$ open.

Setting $q_{E_U}(s) = \pi_{E_U}(s)$, hence, for $r \in \pi_U(E_U)$, we get

$$r^2 = q_{\pi_{U(E_U)}}(r).1_{\tau_U}$$

For a quadratic differential triad $qdT_X \equiv ((\mathcal{A}_X , q_{\mathcal{A}_X}), \partial_X , (\Omega_X, q_{\Omega_X})$ over $(X, \mathcal{A}_X)$, the Clifford differential triad relative is represented by the triplet

$$\left(C\left(\mathcal{A}_X, q_{\mathcal{A}_X}\right), \partial_X^c , C\left(\Omega_X, q_{\Omega_X}\right)\right) \tag{3.12}$$

such that

(a) $\phi: \left(\mathcal{A}_X, q_{\mathcal{A}_X}\right) \to \mathcal{A}_X$ and $\Psi: \left(\Omega_X, q_{\Omega_X}\right) \to \mathcal{A}_X$ are linear maps satisfying

$$\begin{cases}\phi(\alpha)\phi(\alpha) = -q_{\mathcal{A}_U}(\alpha).1, & \alpha \in \mathcal{A}_U \\ \psi(\alpha)\psi(\alpha) = -q_{\Omega_U}(s).1, & s \in \Omega_U\end{cases} \tag{3.13}$$

(b) Φ and ψ extend uniquely to $\hat{\phi}: \left(\mathcal{A}_X, q_{\mathcal{A}_X}\right) \to \mathcal{A}_X$ and $\hat{\psi}: C\left(\Omega_X, q_{\Omega_X}\right) \to \Omega_X$

We set

$$CdT_X = \left(C\left(\mathcal{A}_X, q_{\mathcal{A}_X}\right), \partial_X^c , C\left(\Omega_X, q_{\Omega_X}\right)\right) \tag{3.14}$$

where *C* designs the functorial morphism which transforms a quadratic differential triad qdT_X to a Clifford differential triad CdT_X and d_X^c satisfies the Leibniz (product) rule as given in (1.7). For convenience , we write

$$CdT_X = \left(C(\mathcal{A}_X), \partial_X^c , C(\Omega_X)\right) \tag{3.15}$$

By Considering $CT_{iX} = \left(C(\mathcal{A}_{iX}), \partial_{iX}^c , C(\Omega_{iX})\right)$ and $CT_{jX} = \left(C\left(\mathcal{A}_{jX}\right), \partial_{jX}^c , C\left(\Omega_{jX}\right)\right)$ as two Clifford differential triads, the morphism from CT_{iX} to CT_{jX} is the triplet

$$\left(Ch_{\mathcal{A}_X}^{ij}, \partial_X^{Cij}, Ch_{\Omega_X}^{ij} \right) \tag{3.16}$$

where $Ch_{\mathcal{A}_X}^{ij} \in Hom_{\mathbb{K}_X}\left(C(\mathcal{A}_{iX}), C\left(\mathcal{A}_{jX}\right)\right)$ and $Ch_{\Omega_X}^{ij} \in Hom_{\mathbb{K}_X}\left(C(\Omega_{iX}), C\left(\Omega_{jX}\right)\right)$ are assumed to be continuous maps and ∂_X^{cij} satisfies the Leibniz (product) rule which verifies for any open *U* in *X,* the relation

$$\partial_X^{cij}\left(Ch_{\mathcal{A}_X}^{ij}\right) = Ch_{\Omega_X}^{ij} \tag{3.17}$$

From the above concepts, it follows that

$$Ch_{\Omega_X}^{ij} \circ \partial_{IX}^c = \partial_{jX}^c \circ Ch_{\mathcal{A}_X}^{ij} \tag{3.18}$$

so that we write

$$d_X^{ij}(Ch_{\mathcal{A}_X}^{ij})|_{\{a_i\}} = \left(Ch_{\Omega_X}^{ij} \circ \partial_{iX}^{c}\right)(a_i) = \left(\partial_{jX}^{c} \circ Ch_{\mathcal{A}_X}^{ij}\right)(a_i) \quad (3.19)$$

for any $a_i \in C(\mathcal{A}_{iX})$. Since the category of Clifford differential triads is denoted by *CDiffT* , clearly the mapping $F: QDiffT \to CDiffT$ behaves nicely as a differential triad functor. Consequently, the mapping $C: DiffT \to CDiffT$ is the Clifford differential triad functor regarding the following composition relation

$$F\ o\ Q = C \quad (3.20)$$

From which one writes

$$C\left(\left(\mathcal{A}_X, q_{\mathcal{A}_X}\right), \partial_X, \left(\Omega_X^1, q_{\Omega_X^1}\right)\right) = \left(C\left(\mathcal{A}_X, q_{\mathcal{A}_X}\right), \partial_X^c, C\left(\Omega_X^1, q_{\Omega_X^1}\right)\right) \quad (3.21)$$

By fixing a common basis topological space *X* ,then we design the categories of differential triads, quadratic differential triads and Clifford differential triads, respectively, by $DiffT_X$, Q$DiffT_X$, C$DiffT_X$. A complex of free left (resp. right) $C(\mathcal{A})$-modules is a sequence of left (resp. right) the following $C(\mathcal{A})$-homomorphisms

$$C(\Omega^*) \equiv \cdots \xrightarrow{d^{c,i-1}} C(\Omega^i) \xrightarrow{d^{c,i}} C(\Omega^{i+1}) \xrightarrow{d^{c,i+1}} \ldots \quad (3.22)$$

between left (resp. right) $C(\mathcal{A})$-modules $C(\Omega^i)$ and $C(\Omega^{i+1})$ which satisfy, for any open *U* in *X*

$$Im\ d^{c,i-1}(U) \subseteq Ker\ d^{c,i}(U), i.e., d_U^{c,i} o d_U^{c,i-1} = 0_U \quad \forall i \in \mathbb{Z} \quad (3.23)$$

Regarding the above composition (3.23), where by definition

$$d^{c,0} = \partial^c, d^{c,i} = d^c \quad (3.24)$$

for any $i \geq 1$, where the symbol *d* designs the differentia*l* $\mathcal{A}$-homomorphism**.**

If we specify the order of the set Ω of differential forms (sheaf of differential $\mathcal{A}$-modules) by setting

$$C(\Omega^0) = C(\mathcal{A}),\ C(\Omega^i) \equiv C(\Omega^1)^i = C(\wedge^i\Omega^1) \quad (3.25)$$

where $\wedge \equiv \wedge_{\mathcal{A}}$ be the exterior (the skew symmetric homological tensor) product, for any

$i \geq 1$. We have, more explicitly

$$C(\Omega^1) = C(\mathcal{A}) \wedge C(\Omega)\ , C(\Omega^2) = C(\mathcal{A}) \wedge C(\Omega^1) \wedge C(\Omega^1) \quad (3.26)$$

A complex of free left (resp. right) $C(\mathcal{A})$-modules denoted by

$$C(\Omega_*) \equiv \ldots \xrightarrow{\int_{i+2}} C(\Omega_{i+1}) \xrightarrow{\int_{i+1}} C(\Omega_i) \xrightarrow{\int_i} \ldots \quad (3.27)$$

is a sequence of left (resp. right) $C(\mathcal{A})$-homomorphisms $\int_{i+1} : C(\Omega_{i+1}) \to C(\Omega_i)$ between left (resp. right) $C(\mathcal{A})$-modules which satisfy, for any open *U* in *X*

$$Im\int_{i+1,U} \subseteq ker\int_{i,U}\ \text{, i.e., } \int_{i,U} o \int_{i+1,U} = 0_U \equiv 0 \quad (3.28)$$

for all $i \in \mathbb{Z}$, where the symbol $\int$ designs the integral $C(\mathcal{A})$-homomorphism. By replacing $C(\Omega^i)$ by CdT^i and $C(\Omega_i)$ by $C(inT_i)$, we obtain , respectively the complexe :

$$C(\mathrm{dT}^*) \equiv \cdots \xrightarrow{mdT^{c,i-1}} C(dT^i) \xrightarrow{mdT^{c,i}} C(dT^{i+1}) \xrightarrow{mdT^{c,i+1}} \ldots$$

And

$$C(inT_*\ ) \equiv \ldots \xleftarrow{m\int inT_i} C(in\int T_i) \xleftarrow{m\int inT_{i+1}} C(inT_{i+1}) \xleftarrow{m\int inT_{i+2}} \ldots$$

We set $C(dT^i) = C(in\int T_i)$ within $C(\mathcal{A})$-isomorphism, and $mdT^{c,i}$ and $m\int inT_{i+1}$ are morphisms, respectively, of Clifford differential triads and of Clifford integral triads, for all $i \in \mathbb{Z}$. In other words, we have (see[15])

$$m\int inT_{i+1} \, o \, mdT^{c,i} = id_{C(dT^i)} \tag{3.29}$$

4. Applications

We suggest some physics applications of Clifford differential triads by revisiting special relativity when we take into consideration a natural algebraic concept alternative to the Minkowski space-time. The two algebras considered are

$(A)_1$ $\mathcal{A}_{\mathbb{R}}$ is the real underlying of the $\mathbb{R}$ −algebra sheaf $\mathcal{A}$;

$(A)_2$ $Cl_{2,0}(\mathcal{A}_{\mathbb{R}})$ is the 2 −dimensional Clifford $\mathcal{A}_{\mathbb{R}}$ −algebra.

Our approach consists to replace the unit imaginary $i = \sqrt{-1}$ by an element e_1e_2 of $Cl_{2,0}(\mathcal{A}_{\mathbb{R}}(U))$, where (e_1, e_2) is an $\mathcal{A}_{\mathbb{R}}(U)$ −basis of $\mathcal{A}^2_{\mathbb{R}}(U)$. In this case, an $\mathcal{A}_{\mathbb{R}}(U)$ −basis of $Cl_{2,0}(\mathcal{A}_{\mathbb{R}}(U))$ is

$$(I, e_1, e_2, e_1e_2) \equiv (I, e_1, e_2, l) \tag{4.1}$$

where $e_1^2 = e_2^2 = I$ and $e_1e_2 = -e_2e_1$, from which we clearly obtain

$$l^2 = (e_1e_2)^2 = e_1e_2e_1e_2 = -e_1e_1e_2e_2 = -e_1^2e_2^2 = -I \tag{4.2}$$

and l is an alternative of $i = \sqrt{-1}$. An element μ in $Cl_{2,0}(\mathcal{A}_{\mathbb{R}})$ is written as follows

$$\mu = Ia + r_1e_1 + r_2e_2 + lb \tag{4.3}$$

where $a \in \mathcal{A}_{\mathbb{R}}$, $r_1e_2 + r_2e_2 \in \mathcal{A}^2_{\mathbb{R}}$, $b \in \mathcal{A}_{\mathbb{R}}$ and $lb \in \Lambda^2\mathcal{A}_{\mathbb{R}}$, i.e., we set

$$Cl_{2,0}(\mathcal{A}_{\mathbb{R}}) = \mathcal{A}_{\mathbb{R}} \oplus \mathcal{A}^2_{\mathbb{R}} \oplus \Lambda^2\mathcal{A}_{\mathbb{R}} = \Lambda^0\mathcal{A}_{\mathbb{R}} \oplus \Lambda^1\mathcal{A}_{\mathbb{R}} \oplus \Lambda^2\mathcal{A}_{\mathbb{R}} \tag{4.4}$$

In short, one writes within $\mathcal{A}_{\mathbb{R}}$ −isomorphism

$$Cl_{2,0}(\mathcal{A}_{\mathbb{R}}) = \Lambda\mathcal{A}_{\mathbb{R}} \tag{4.5}$$

where $\Lambda\mathcal{A}_{\mathbb{R}}$ is the exterior $\mathcal{A}_{\mathbb{R}}$ −algebra of $\mathcal{A}_{\mathbb{R}}$.

For two vectors given by $\mu = r_1e_1 + r_2e_2$ and $\mu' = r_1'e_1 + r_2'e_2$ in $Cl_{2,0}(\mathcal{A}_{\mathbb{R}})$, one can check that their product called the Clifford (or geometric) product, is determined as follows

$$\mu\,\mu' = \mu.\mu' + \mu \wedge \mu' \tag{4.6}$$

where $\mu.\mu' = (r_1r_1' + r_2r_2')$ and $\mu \wedge \mu' = (r_1r_2' - r_2r_1')\, l$.

The distance measure or metric over the space $Cl_{2,0}(\mathcal{A}_{\mathbb{R}})$ is $\mu.\mu'$. Considering the following map

$$c_X: (\mathcal{A}_{\mathbb{R}} \oplus \mathcal{A}_{\mathbb{R}})_X \to Cl_{2,0}(\mathcal{A}_{\mathbb{R}})_X$$

defined, for any open U in X, by

$$c_U(\xi_\alpha) \equiv c_U(\xi_1, \xi_2) := (\eta_0, \eta_1, \eta_2, \eta_3) \equiv \eta_i \tag{4.7}$$

with $i = o, 1, 2, 3$ such that $\eta_0 = a$, $\eta_1 = r_1$, $\eta_2 = r_2$, $\eta_3 = b \in (\mathcal{A}_{\mathbb{R}})_U$.

We observe that if (e_1, e_2) is $(\mathcal{A}_{\mathbb{R}})_U$ −basis of $(\mathcal{A}_{\mathbb{R}} \oplus \mathcal{A}_{\mathbb{R}})_U$, then (I, e_1, e_2, e_1e_2) is $(\mathcal{A}_{\mathbb{R}})_U$ −basis of $Cl_{2,0}(\mathcal{A}_{\mathbb{R}})_U$.

Using differential triad notation, we get

$$qdT_X := ((\mathcal{A}_{\mathbb{R}} \oplus \mathcal{A}_{\mathbb{R}})_X, \partial_X, \Omega^1(\mathcal{A}_{\mathbb{R}} \oplus \mathcal{A}_{\mathbb{R}})_X) \tag{4.8}$$

and

$$qdT_X := (Cl_{2,0}(\mathcal{A}_{\mathbb{R}})_X, \partial_X^C, \Omega^1(Cl_{2,0}(\mathcal{A}_{\mathbb{R}})_X)) \tag{4.9}$$

where ∂_X^C designs the Clifford differentials.

Considering the 2- real Euclidean differential triad, for any open U in X, we have

$$d_U(r_1, r_2) = (d_U r_1, d_U r_2) \equiv (dr_1, dr_2) = d_{r_j},\ j = 1,2 \tag{4.10}$$

so that the Pythagorean distance measure (or metric) is written as

$$dS^2 = dr_1^2 + dr_2^2$$
$$dS^2 = (dr_1, dr_2)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}dr_1\\ dr_2\end{pmatrix}$$
$$dS^2 \equiv \delta_{jk} dr_j dr_k, \qquad j,k = 1,2$$

By setting $\delta = det\delta_{jk} = 1$, then the motion of a particle of mass M is governed by the following 2-velocity and 2-acceleration

$$\begin{cases} v_j = \frac{dr_j}{d\tau} \\ \gamma_j = \frac{dv_j}{d\tau} \end{cases} \tag{4.11}$$

where $\frac{d}{d\tau}$ designs the proper-time derivative in $\mathbb{R}^{\mathcal{A}} \oplus \mathbb{R}^{\mathcal{A}}$.

In partial derivatives notation, we write

$$\partial_j = \frac{\partial}{\partial r_j} = \partial_{r_j} \quad j = 1,\ 2 \tag{4.12}$$

In the integral triad, we write

$$\int_{\mathcal{A}_{\mathbb{R}}^2} dr \equiv \int d^2 r = \int \sqrt{\delta} d^2 r \tag{4.13}$$

For n-dimensional Euclidean space, the above expression is written as

$$\int_{\mathcal{A}_{\mathbb{R}}^n} dr \equiv \int d^n r = \int \sqrt{\delta} d^n r$$

and v_j and γ_j, (with $j = 1, 2, ..., n$) are the *n*-velocity and the *n*-acceleration of the particle.

In Clifford algebra differential triads notation, for any open *U* in *X*

$$d_U^C R \equiv d_U^C (r + lct) \equiv d_U r + lc d_U t \equiv dr + lcdt \tag{4.14}$$

and

$$d_U^C R^2 = d_U r^2 - c^2 d_U t^2 \tag{4.15}$$

For τ as the proper-time of the particle and using initial condition (assuming that $d_U r^2 = 0$) , expressions (4.14) and (4.15) become

$$d_U^C R_0 = dr_0 + lcd\tau,\ d_U^C R_o^2 = -c^2 d_U \tau^2 \tag{4.16}$$

It follows that, for a space-time interval , we have $d_U^C R_0^2 = d_U^C R^2$ so that

$$-c^2 d_U \tau^2 = d_U r^2 - c^2 d_U \tau^2 \tag{4.17}$$

In other terms, for $d_U r = v d_U \tau$ expression (4.17) becomes

$$c^2 d_U \tau^2 = c^2 d_U \tau^2 \left(1 - \frac{v^2}{c^2}\right) \tag{4.18}$$

Setting

$$\Gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{4.19}$$

Then we get the famous dilatation formula

$$d\tau \equiv d_U \tau = \Gamma d_U \tau \tag{4.20}$$

In terms of proper-time, the proper velocity becomes

$$V = \frac{d^C R}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau} + lc\frac{dt}{d\tau} = v\frac{dt}{d\tau} + lc\frac{dt}{d\tau} = \Gamma(v + lc) \tag{4.21}$$

From (4.21), we easily obtain

$$V^2 = \left(\frac{d^C R}{d\tau}\right)^2 = \Gamma^2(v + lc)^2 = \Gamma^2(v^2 - c^2),\ vlc = -lcv \tag{4.22}$$

where $V^2 = -c^2$ and the fact that l anticommutes with each component of v and $l^2 = -1$.
If M designs a massless particle, then the linear momentum is

$$P = \Gamma(Mv + Mlc)$$

so that

$$P^2 = -M^2V^2 \tag{4.23}$$

By setting

$$p = \Gamma Mv \text{ and } E = \Gamma Mc^2 \tag{4.24}$$

then p and E are respectively the relativistic linear momentum and the total energy. It follows that

$$P = p + \frac{E}{c}\, l \tag{4.25}$$

If we set $V_\mu = \frac{d^C R_\mu}{d\tau}$, then the motion of free particle with mass M is governed by the equation

$$\gamma_\mu = \frac{V_\mu}{d\tau} = 0$$
$$\text{With } \mu = 1,2,3$$

where V_μ is the *4*-velocity and γ_μ the *4*-acceleration.

We can also write

$$V = \frac{d^C R}{d\tau} \quad \text{and} \quad \gamma = \frac{d^C V}{d\tau} \tag{4.26}$$

where $\frac{d^C}{d\tau}$ designs the proper-time derivative in $Cl_{2,0}(\mathcal{A}_{\mathbb{R}})$.

Using partial derivatives, we obtain :

$$\partial_\mu^C \equiv \frac{\partial^C}{\partial \eta_\mu} \equiv \partial_{\eta_\mu}^C = l\partial_t + e_1\partial_{r_1} + e_1\partial_{r_2} = l\partial_t + \nabla \tag{4.27}$$

where $\nabla = e_1\partial_{r_1} + e_1\partial_{r_2}$ is the space gradient operator and ∂_μ^C is the space-time gradient operator. Consequently, we obtain

$$(\partial_\mu^C)^2 = l^2\partial_t^2 + \nabla^2 + l\partial_t\nabla + \nabla l\partial_t = -\partial_t^2 + \nabla^2 + l\partial_t\nabla - l\nabla\partial_t = -\partial_t^2 + \nabla^2 + l\partial_t\nabla - l\partial_t\nabla.$$

In other terms we have

$$(\partial_\mu^C)^2 = -\partial_t^2 + \nabla^2 \tag{4.28}$$

where l anticommutes with each component of ∇ and $l^2 = -1$.

Thus , we set

$$\square_\eta^C = -(\partial_{\eta_\mu}^C)^2 = \nabla^2 - \partial_t^2 \tag{4.29}$$

and say that $\square_\eta^C$ is the Clifford d'Alembertian operator on scalars (or on multivectors).

For a (real) free massive scalar (or multivector) field Ψ, then we set

$$\partial_\eta^C\Psi = lM\Psi \text{ or } \partial_\eta^C\Psi = lIM\Psi \tag{4.30}$$

where I is the identity matrix and observe that

$$(\partial_\eta^C)^2 = -M^2 \quad \Longmapsto \quad (-\partial_t^2 + \nabla^2)\Psi = -M^2\Psi \quad \Longmapsto \quad (\nabla^2 - \partial_t^2)\Psi = -M^2\Psi \tag{4.31}$$

Using (4.29), we obtain the Clifford Klein-Gordon equation

$$(\Box_\eta^C - M^2)\Psi = 0 \tag{4.32}$$

The Pythagorean distance measure (or Riemann metric) in $Cl_{2,0}(\mathcal{A}_\mathbb{R})$ is given by

$$dS^2 = dr^2 - c^2dt^2 = dS^2 = dr_1^2 + dr_2^2 - c^2dt^2$$

Using matrix and tensor notations, we obtain respectively

$$ds^2 = (dr_1, dr_2, cdt)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}dr_1\\ dr_2\\ cdt\end{pmatrix}$$

And $$dS^2 = g^{\mu\varepsilon}d\eta_\mu d\eta_\varepsilon, \quad \mu,\varepsilon = 1,2,3$$

with $g = detg_{\mu\varepsilon} = -1$. Using (4.29), thus it is clear that

$$\Box_\eta^C = g^{\mu\varepsilon}\nabla_\mu\nabla_\varepsilon - \partial_t^2 \tag{4.33}$$

and say that $\Box_\eta^C$ is the Clifford d'Alambertian operator associated to the Riemann metric $g^{\mu\varepsilon}$.
In integral triad form in $Cl_{2,0}(\mathcal{A}_\mathbb{R})$, we write

$$\int_{Cl_{2,0}(\mathcal{A}_\mathbb{R})} dr \equiv \int d^C\eta_\mu = \int \sqrt{g}d^3\eta$$

5. Conclusion

We have studied differential triads as basic notions through which fundamental concepts of abstract differential geometry were constructed. We have constructed Clifford differential triads with the help of quadratic differential triads. We have come up with category of Clifford differential triads determined through the category of differential triads or category of quadratic differential triads. Some physics applications in special relativity were suggested.

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