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RESEARCH ARTICLE

A COMMON FIXED POINT THEOREM FOR COMPATIBLE MAPPINGS IN S METRIC SPACE

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ABSTRACT

The aim of this paper is to present a common fixed point theorem in a S metric space which extends the results of P.C. Lohani and V.H. Bhadshah using the weaker conditions such as Weakly compatible and Associated sequence. Very recently Sedghi ,Shobe and Aliouche[14] introduced S –metric space as a generalization of metric space and several researchers have proved fixed point theorems for self maps of such spaces.

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INTRODUCTION

G.Jungck gave a common fixed point theorem for commuting mapping maps, which generalizes the Banach's fixed point theorem. This result was further generalized and extended in various ways by many authors. S.Sessa [5] defined weak commutativity and proved common fixed point theorems for weakly commuting maps. Further G. Jungck [1] initiated the concept of compatible maps which is weaker than weakly commuting maps. After wards Jungck and Rhoades [4] defined weaker class of maps known as weakly compatible maps. *D*-metric spaces* by Sedghi,Shobe and Zhou [13] and most recently *S-metric spaces* by Sedghi, Shobe and Aliouche [24] were introduced .Also several fixed point theorems for self maps of S-metric spaces were established in recent years. For examples, see [11],[12],[19],[24] and [25].

In this we deal with S-metric spaces defined in [24] (Definition 2.1) as follows

The purpose of this paper is to prove a common fixed point theorem for four self maps using weakly compatible mappings.

Definitions and Preliminaries

1.1 Definition In this section we present some preliminary results needed for our purpose. We begin with **Definition** ([4]). Let X be a non empty set. An *S-metric* on X is a function S: $X^3 \rightarrow (0, \infty)$ that satisfies the conditions given below for $x, y, z, w \in X$

(i) $S(x, y, z) \geq 0$

(ii) $S(x, y, z) = 0 \text{ if and only if } x = y = z$
and

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(iii) $S(x,y,z) \leq S(x,x,w) + S(y,y,w) + (z,z,w)$

The pair (X, S) is called an *S-metric space.*

If (X, S) is an S-metric space it is shown in ([4], Lemma 2.5) that (2.2) $S(x, x, y) = (y, y, x)$ for all $x, y \in X$ and as a consequence of (iii) of

1.2 Definition 2.1 and (2.2) we have (2.3) $S(x,x,y) \leq 2.S(x,y,z) + S(x,y,z)$for $x, y, z \in X$

A Sequence $\{x_n\}$ in (X, S) is said to

(i) *Converge* to x if to each $\varepsilon > 0$ there is a natural number n_0 such that $(x_n, x_n, x) < \varepsilon$ for all $n \geq n_0$ and
(ii) be a *Cauchy Sequence* if to each $\varepsilon > 0$ there is a natural number n_0 such that $S(x_n, x_n, x_m) < \varepsilon$ for all $m \geq n0$, $n \geq n0$.It is shown in ([4], Lemma 2.10 and Lemma 2.11) that in an S-metric space(X, S) if $\{ xn \}$ converges to x then x is unique and that $\{ xn \}$is a Cauchy Sequence. An S-metric space is said to be *complete* if every Cauchy Sequence in it converges to a point in X.It is easy to prove :(2.4) If $\{x_n\}$ and $\{y_n\}$ in X are converging respectively to x and y in X then $\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x,x,y)$ ([14],Lemma 2.12)

1.3 Definition . If f and g are mappings from a S metric space (X,S) into itself are called weakly commuting mappings on X ,if $S(fgx, fgx, gfx) \leq S(fx, fx, gx)$ for all x in X.

1.4 Definition: Two self maps f and g of a S metric space (X,S) are said to be compatible mappings if $\lim_{n\to\infty} S(fgx_n, fgx_n, gfx_n) = 0$ Whenever $< x_n >$ is a sequence in X such that $\lim_{n\to\infty} f\, x_n = \lim_{n\to\infty} g\, x_n$ = t for some t ∈ X .Clearly commuting mappings are weakly commuting ,but the converse is not necessarily true.

1.5 Definition : Two self maps f and g of a S metric space (X,S) are said to be weakly compatible if they commute at their coincidence point that is if $fu = gu$ for u ∈ X then $fgu = gfu$.
It is clear that every compatible pair is weakly compatible but its converse need not be true.
P.C Lohani and V H Badshah proved the following theorem.

Theorem (A)

Let P, Q, f and g be self mappings from a complete S metric space (X,S) into itself satisfying the following conditions

f(x) ⊂ Q(x) and g(x) ⊂ P(x) ……………………………………….(a)

$S(fx,fx,gy) \leq \frac{\alpha\, S(Qy,Qy,gy)[1+S(Px,Px,fx)]}{1+S(Px,Px,Qy)} + \beta S(Px,Px,Qy)$ for all x,y in X where $\alpha, \beta \geq 0, \alpha+\beta < 1$ …………………….(b)

One of P, Q ,f and g is continuous ……………………………………….(c)

Pair (f, P) and (g, Q) are compatible on X ……………………………………….(d)

Then P, Q, f and g have a unique common fixed point in X.

Associated sequence Suppose P,Q ,f and g are self maps of a S metric space (X,S) satisfying the condition (1).Then for an arbitrary $x_0 \in X$ such that f x_0 =Q x_1 and for this point x_1, there exist a point x_2 in X such that g x_1 =P x_2 and so on. Proceeding in the similar manner, we can define a sequence $< y_n >$ in X such that y_{2n}= f x_{2n}= Q y_{2n+1} and y_{2n+1} = P x_{2n+2} = g x_{2n+1} for $n \geq 0$.

We shall call this sequence as an "Associated sequence of x_0" relative to the four self maps P, Q, f and g.

Lemma: Let P, Q , f and g be self mappings from a complete S metric space (X,S) into itself satisfying the condition (a) and(b)

$f(x) \subset Q(x)$ *and* $g(x) \subset P(x)$ ……………………………………...(a)

$S(fx,fx,gy) \leq \frac{\alpha S(Qy,Qy,gy)[1+S(Px,Px,fx)]}{1+S(Px,Px,Qy)} + \beta S(Px,Px,Qy)$ for all x,y in X where $\alpha, \beta \geq 0, \alpha+\beta < 1$ ……………………..(b)

Then the associated sequence $<y_n>$ relative to four self maps is a Cauchy sequence in X.

Proof

From (2), we have
$$S(y_{2n}, y_{2n}, y_{2n+1}) = S(fx_{2n}, fx_{2n}, g\, x_{2n+1}) \leq \frac{\alpha\, S(Q\, x_{2n+1}, Q\, x_{2n+1}, gx_{2n+1})[1+S(Px_{2n}, Px_{2n}, fx_{2n})]}{1+S(Px_{2n}, Px_{2n}, Qy_{2n+1})} + \beta\, S(Px_{2n}, Px_{2n}, Qy_{2n+1})$$

$= \frac{\alpha S(y_{2n},y_{2n},y_{2n+1})[1+S((y_{2n-1},y_{2n-1},y_{2n+1})]}{1+S(y_{2n-1},y_{2n-1},y_{2n})} + \beta S(y_{2n-1},y_{2n-1},y_{2n})$

$= \alpha S(y_{2n},y_{2n},y_{2n+1}) + \beta S(y_{2n-1},y_{2n-1},y_{2n})$

$S(y_{2n},y_{2n},y_{2n+1}) \le \frac{\beta}{(1-\alpha)} S(y_{2n-1},y_{2n-1},y_{2n})$

$S(y_{2n},y_{2n},y_{2n+1}) \le h\, S(y_{2n-1},y_{2n-1},y_{2n})$ where $h = \frac{\beta}{(1-\alpha)}$

Now

$S(y_n,y_n,y_{n+1}) \le hS(y_{n-1},y_{n-1},y_n)$

$\le h^2 S(y_{n-2},y_{n-2},y_{n-1})$

$\le h^n S(y_0,y_0,y_1)$

For every integer p>0 we get

$S(y_n,y_n,y_{n+p}) \le S(y_n,y_n,y_{n+1}) + S(y_{n+1},y_{n+1},y_{n+2}) + --------------- S(y_{n+p-1},y_{n+p-1},y_{n+p})$

$\le h^n S(y_0,y_0,y_1) + h^{n+1} S(y_0,y_0,y_1) \pm --- h^{n+p-1} S(y_0,y_0,y_1)$

$= h^n(1 + h + ----- h^{p-1}) S(y_0,y_0,y_1)$

Since h<1 ,$h^n \to 0$ as $n \to \infty$ so that $S(y_n,y_n,y_{n+p}) \to 0$

This shows that the sequence $<y_n>$ is a Cauchy sequence in X and since X is a complete S metric space ;it converges to a limit say z ∈X

The converse of the lemma is not true that is P,Q ,f and g are self maps of a S metric space (X,S) satisfying (a) and (b) even if for $x_0 \in X$ and for associated sequence of x_0 converges the S metric space(X,S) need not be complete.

Example: Let X =(-1,1) with d(x, y) = $|x - y|$

$$fx = gx = \begin{cases} \frac{1}{5}\ if\ -1 < x < \frac{1}{6} \\ \frac{1}{6}\ if\ \frac{1}{6} \le x < 1 \end{cases}$$

$$Px = \begin{cases} \frac{1}{5}\ if\ -1 < x < \frac{1}{6} \\ \frac{6x+5}{36}\ if\ \frac{1}{6} \le x < 1 \end{cases} \quad Qx = \begin{cases} \frac{1}{5}\ if\ -1 < x < \frac{1}{6} \\ \frac{1}{3} - x\ if\ \frac{1}{6} \le x < 1 \end{cases}$$

Then $(X) = g(X) = \{\frac{1}{5},\frac{1}{6}\}$,while $P(X) = \{\frac{1}{5} \cup [\frac{1}{6},\frac{11}{36})\}$, $Q(X) = \{\frac{1}{5} \cup [\frac{1}{6},\frac{-2}{3})\}$ so that $f(x) \subset Q(x)$ and $g(x) \subset P(x)$ proving the condition (a). Clearly (X, d) is not a complete metric space. It is easy to prove that the associated sequence $f x_0, g x_1, f x_2, g x_3, ---------, f x_{2n}, g x_{2n+1}$.., converges to $\frac{1}{5}\ if\ -1 < x < \frac{1}{6}$ or $\frac{1}{6} \le x < 1$, the associated sequence is converges to $\frac{1}{6}$ .Now we prove our theorem.

Theorem (B)

Let P,Q, f and g be self mappings from a complete S metric space (X,S) into itself satisfying the following conditions

$f(x) \subset Q(x)$ and $g(x) \subset P(x)$ ……………………………….(e)

$S(fx,fx,gy) \le \frac{\alpha S(Qy,Qy,gy)[1+S(Px,Px,fx)]}{1+S(Px,Px,Qy)} + \beta S(Px,Px,Qy)$ for all x, y in X where $\alpha, \beta \ge 0, \alpha + \beta < 1$ ……………….(f′)

and the conditions .The pairs (f, P)and (g, Q) are weakly compatible and One of P, Q, f and g is continuous also the associated sequence relative to four self maps P, Q, f and g such that the sequence $f x_0, g x_1, f x_2, g x_3, ---------, f x_{2n}, g x_{2n+1}$ converges to z ∈ X as $n \to \infty$ ------(g′).Then P, Q, f and g have a unique common fixed point z in X

Proof:

From the condition (3) $f\,x_0\,, g\,x_1, f\,x_2\,, g\,x_3, ----------, f\,x_{2n}, g\,x_{2n+1}$ converges to z ∈ $X\ as\ n\ \rightarrow \infty$

Since $f(x) \subset Q(x)$ then there exists u ∈ X such that $z = QU$ we prove that $Q\,u = g\,u = z$.

we consider

$$S(gu, gu, z) = S(z, z, gu) = S(fx_{2n}, fx_{2n}, gu) \le \lim_{n\to\infty} \frac{\alpha S(Qu, Qu, gu)[\,1 + S(Px2n, Px2n, fx2n)]}{1 + S(Px2n, Px2n, Qu)}$$

$$+\,\beta\, S(Px, Px, Qu\,)$$

$$= \frac{\alpha S(z, z, gu)[\,1 + S(z, z, z)]}{1 + S(z, z, z)} + \beta\, S(z, z, z\,)$$

$$= \alpha S(z, z, gu)$$

$$S(z, z, gu) \le \alpha\, S(z, z, gu)$$

$(1—\alpha)S(z, z, gu) \le 0$ which implies that $z = g\,u$

Therefore $Q\,u = g\,u = z$

Since (Q ,g) is weakly compatible $Qg\,u = gQu$

Which implies $Qz = gz$

and $g(x) \subset P(x)$ there exists v ∈ X such that $z = P\,v$

we solve $fv = Pv$

Consider $S(fv, fv, gx_{2n+1}) \le \frac{\alpha\, S(Qx_{2n+1}, Qx_{2n+1}, gx_{2n+1})[\,1+S(Pv,Pv,fv)]}{1+S(Pv,Pv,Qx_{2n+1})} + \beta S(Pv, Pv, Qx_{2n+1})$

$S(fv, fv, z) \le 0$

Which implies that $fv = z$

Since $fv = Pv = z$ and (f, P) is weakly compatible $fPv = Pfv$ which implies that $fz = Pz$.

Now consider $S(fz, fz, z)$=$\lim_{n\to\infty} S(fz, fz, gu)$

$$\le \lim_{n\to\infty} \frac{\alpha S(Qu,Qu,gu)[\,1+S(Pz,Pz,fz)]}{1+S(Pz,Pz,Qu)} + \beta S(Pz, Pz, Qu)$$

$$= \beta S(fz, fz, z)$$

Since $\alpha + \beta < 1$

$S(fz, fz, z) = 0$

Which implies that $fz = z$

Which implies that $fz = Pz$

Therefore z is common fixed point of f and P

Again we consider

$$S(z, z, gz) = S(fz, fz, gz) \le \frac{\alpha\, S(Qz, Qz, gz)(1 + S(Pz, Pz, fz)}{1 + S(Pz, Pz, Qz)} + \beta S(Pz, Pz, Qz)$$

$$= \beta S(z, z, gz)$$

$Which\ implies\ S(z,z,gz)) \leq \beta S(z,z,gz)$

Since $\beta \geq 0, \quad \alpha+\beta<1$

$S(z,z,gz)=0$

Thus $gz = z$

Therefore $z = Q\,z = g\,z$ then z is a common fixed point of g and Q

This gives $S(fz,fz,z)\,) \leq \beta S(fz,fz,z)$

Since $\beta \geq 0, \quad \alpha+\beta<1$

$S(fz,fz,z)=0$

Thus $fz = z$

Therefore $f\,z = Pz = z = Q\,u$

Yhis shows that z is a common fixed point of P and f

Therefore $Pz = Qz = fz = gz = z$ showing that z is a common fixed point of P,Q, f and g.

Remark: Theorem (B) is a generalization of Theorem(A) by virtue of the weaker conditions such as weakly compatibility of the pairs (f, P) and (g, Q) in place of compatibility; and associated sequence relative to four self maps P, Q, f and g in place of the complete metric space.

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