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RESEARCH ARTICLE

EVALUATION OF AN ATTRACTIVE INTEGER TRIPLE

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ABSTRACT

We search for non-zero distinct integer *(a, b, c)* such that the sum of any two of them is a cubical integer.

INTRODUCTION

Diophantine equations are numerously rich because of its variety. Diophantine problems were first introduced by Diophantus of Alexandria who studied this topic in the Algebra. The theory of Diophantine equations is a treasure house in which the search for many hidden relation and properties among numbers from a treasure hunt. In fact Diophantine problems dominated most of the celebrated unsolved mathematical problems. Certain Diophantine problems come from physical problems or from immediate Mathematical generalizations and others come from geometry in a variety of ways. Certain Diophantine problems are neither trivial nor difficult to analyse (James Matteson, 1888; Titu andreescu and Dorin Andrica, 2002). Also, one may refer (Dickson, 2005; Carmichael, 1959; Mordell, 1969; John *et al.,* 1995; Gopalan *et al.,* 2012, 2013 & 2015). In this paper, We search for non-zero distinct integer triple (a, b, c) such that each of the expressions $a+b$, $a+c$, $b+c$, is a cubical integer.

Method of analysis

Let (a, b, c) be three non –zero distinct integers such that

$$a+b=\alpha^3 \tag{1}$$

$$a+c=\beta^3 \tag{2}$$

$$b+c=\gamma^3 \tag{3}$$

$$2(a+b+c)=(\alpha+\beta+\gamma)\delta^2 \tag{4}$$

Solving the system of equations from (1) to (3), we have

$$a=\frac{1}{2}\left(\alpha^3+\beta^3-\gamma^3\right) \tag{5}$$

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$$b = \frac{1}{2}\left(\alpha^3 + \gamma^3 - \beta^3\right) \tag{6}$$

$$c = \frac{1}{2}\left(\beta^3 + \gamma^3 - \alpha^3\right) \tag{7}$$

Adding (5), (6) and (7), we get

$$2(a + b + c) = \left(\alpha^3 + \beta^3 + \gamma^3\right) \tag{8}$$

In view of (4) and (8), we obtain

$$\left(\alpha^3 + \beta^3 + \gamma^3\right) = (\alpha + \beta + \gamma)\delta^2 \tag{9}$$

Introducing the linear transformations

$$\alpha = p + q,\ \beta = p - q,\ \gamma = p \tag{10}$$

where p and q are non-zero parameters in (9), we get

$$p^2 + 2q^2 = \delta^2 \tag{11}$$

By applying four different patterns of solutions to (1), the process of finding triple (a, b, c) such that the sum of any of them is a cubical integer is explained below.

Case i

Consider the general solution to (11) as $p = 2m^2 - n^2,\ q = 2mn,\ \delta = 2m^2 + n^2$ (12)

In view of (12) and (10), we get

$$\alpha = 2m^2 - n^2 + 2mn,\ \beta = 2m^2 - n^2 - 2mn,\ \gamma = 2m^2 - n^2 \tag{13}$$

Substituting (13) in (5),(6) and (7), we obtain

$$a = \frac{1}{2}\left[\left(2m^2 - n^2 + 2mn\right)^3 + \left(2m^2 - n^2 - 2mn\right)^3 - \left(2m^2 - n^2\right)^3\right]$$

$$b = \frac{1}{2}\left[\left(2m^2 - n^2 + 2mn\right)^3 + \left(2m^2 - n^2\right)^3 - \left(2m^2 - n^2 - 2mn\right)^3\right]$$

$$c = \frac{1}{2}\left[\left(2m^2 - n^2 - 2mn\right)^3 + \left(2m^2 - n^2\right)^3 - \left(2m^2 - n^2 + 2mn\right)^3\right]$$

We note that the triple (a, b, c) is integer when m is arbitrary and n is even.

choose $n = 2N$

Thus,

$$a = 4\left[\left(m^2 - 2N^2 + 2mN\right)^3 + \left(m^2 - 2N^2 - 2mN\right)^3 - \left(m^2 - 2N^2\right)^3\right]$$

$$b = 4\left[\left(m^2 - 2N^2 + 2mN\right)^3 + \left(m^2 - 2N^2\right)^3 - \left(m^2 - 2N^2 - 2mN\right)^3\right]$$

$$c = 4\left[\left(m^2 - 2N^2 - 2mN\right)^3 + \left(m^2 - 2N^2\right)^3 - \left(m^2 - 2N^2 + 2mN\right)^3\right]$$

Some numerical examples are presented below:

m	n	a	b	c	$(a+b)$	$(b+c)$	$(c+a)$
2	1	800	928	-864	$(12)^3$	$(4)^3$	$(-4)^3$
1	2	-4060	3844	-6588	$(-6)^3$	$(-14)^3$	$(-22)^3$
3	2	3460	14116	-14108	$(26)^3$	$(2)^3$	$(-22)^3$
4	3	-27680	112864	-112928	$(44)^3$	$(-4)^3$	$(52)^3$
7	4	1299140	1812996	-1773692	$(146)^3$	$(34)^3$	$(-78)^3$

Case ii

Consider the general solution to (11) as

$$p = m^2 - 2n^2 \,,\ q = 2mn,\ \delta = m^2 + 2n^2 \tag{14}$$

In view of (14) and (10), we get

$$\alpha = m^2 - 2n^2 + 2mn \,,\ \beta = m^2 - 2n^2 - 2mn \,, \lambda = m^2 - 2n^2 \tag{15}$$

Substituting (15) in (5), (6) and (7), we obtain

$$a = \frac{1}{2}\left[(m^2 - 2n^2 + 2mn)^3 + (m^2 - 2n^2 - 2mn)^3 - (m^2 - 2n^2)^3\right]$$

$$b = \frac{1}{2}\left[(m^2 - 2n^2 + 2mn)^3 + (m^2 - 2n^2)^3 - (m^2 - 2n^2 - 2mn)^3\right]$$

$$c = \frac{1}{2}\left[(m^2 - 2n^2 - 2mn)^3 + (m^2 - 2n^2)^3 - (m^2 - 2n^2 + 2mn)^3\right]$$

Since our interest is to find the integer triple, we observe that the triple $(a\ ,b\,,c)$ is integer when n is arbitrary and m is even. choose $m = 2M$

Therefore,

$$a = 4\left[(2M^2 - n^2 + 2Mn)^3 + (2M^2 - n^2 + 2Mn)^3 - (2M^2 - n^2)^3\right]$$

$$b = 4\left[(2M^2 - n^2 + 2Mn)^3 + (2M^2 - n^2)^3 - (2M^2 - n^2 - 2Mn)^3\right]$$

$$c = 4\left[(2M^2 - n^2 - 2Mn)^3 + (2M^2 - n^2)^3 - (2M^2 - n^2 + 2Mn)^3\right]$$

Some numerical examples are illustrated below

m	n	a	b	c	$(a+b)$	$(b+c)$	$(c+a)$
2	1	4060	6588	-3844	$(22)^3$	$(4)^3$	$(6)^3$
2	2	6400	7424	-6912	$(24)^3$	$(8)^3$	$(-8)^3$
1	3	-7420	7412	-10156	$(-2)^3$	$(-14)^3$	$(-26)^3$
3	2	59360	81248	-59296	$(52)^3$	$(28)^3$	$(4)^3$
7	4	8377120	12647456	-8236512	$(276)^3$	$(164)^3$	$(52)^3$

Case iii

Rewrite (11) as

$$p^2 + 2q^2 = \delta^2 \times 1 \tag{16}$$

Assume that $\delta = \delta(r, s) = r^2 + 2s^2$ (17)

where r, s are non-zero distinct integers

Replace 1 by

$$1 = \frac{(1 + i2\sqrt{2})(1 - i2\sqrt{2})}{9} \tag{18}$$

Using (17) and (18) in (16), we get

$$p^2 + 2q^2 = \frac{(1 + i2\sqrt{2})(1 - i2\sqrt{2})}{9}(r^2 + 2s^2)^2$$

Expanding the right hand side of the above equation and equating the positive parts on both sides, we obtain

$$(p + iq\sqrt{2}) = \frac{1}{3}(r + is\sqrt{2})^2 (1 + i2\sqrt{2}) \tag{19}$$

Equating the rational and irrational parts, we get

$$p = \frac{1}{3}\left[r^2 - 2s^2 - 8rs\right]$$

$$q = \frac{1}{3}\left[2r^2 - 4s^2 + 2rs\right]$$

Here, the values of p and q are integers when $r = 3R \; and \; s = 3S$

Thus,

$$p = \left[3R^2 - 6S^2 - 24RS\right], \; q = \left[6R^2 - 12S^2 + 6RS\right], \; \delta = 9R^2 + 18S^2, \text{ where } R \neq S \tag{20}$$

using (20) in (10), we get

$$\alpha = 9R^2 - 18S^2 - 18RS \;, \; \beta = -3R^2 + 6S^2 - 30RS \;, \lambda = 3R^2 - 6S^2 - 24RS \tag{21}$$

Substituting (21) in (5), (6) and (7), we obtain

$$a = \frac{1}{2}\left[(9R^2 - 18S^2 - 18RS)^3 + (-3R^2 + 6S^2 - 30RS)^3 - (3R^2 - 6S^2 - 24RS)^3\right]$$

$$b = \frac{1}{2}\left[(9R^2 - 18S^2 - 18RS)^3 + (3R^2 - 6S^2 - 24RS)^3 - (-3R^2 + 6S^2 - 30RS)^3\right]$$

$$c=\frac{1}{2}\left[\left(-3R^2+6S^2-30RS\right)^3+\left(3R^2-6S^2-24RS\right)^3-\left(9R^2-18S^2-18RS\right)^3\right]$$

We find that the triple (a, b, c) is integer when S is arbitrary and R is even.

Choose $R=2T$

Hence, the values of a, b, c satisfying our assumption are given by

$$a=4\left[\left(18T^2-9S^2-18ST\right)^3+\left(-6T^2+3S^2-30ST\right)^3-\left(6T^2-3S^2-24ST\right)^3\right]$$

$$b=4\left[\left(18T^2-9S^2-18ST\right)^3+\left(6T^2-3S^2-24ST\right)^3-\left(-6T^2+3S^2-30ST\right)^3\right]$$

$$c=4\left[\left(-6T^2+3S^2-30ST\right)^3+\left(6T^2-3S^2-24ST\right)^3-\left(18T^2-9S^2-18ST\right)^3\right]$$

Some numerical examples are presented below

T	S	a	b	c	$(a+b)$	$(b+c)$	$(c+a)$
1	2	-629856	-629856	-629856	$(-108)^3$	$(-108)^3$	$(-108)^3$
2	3	-15881292	3068388	-28480572	$-(234)^3$	$(-294)^3$	$(-354)^3$
2	1	-1968300	2125764	-2283228	$(54)^3$	$(-54)^3$	$(-162)^3$
5	4	-1168382880	1167123168	-1599204384	$(-108)^3$	$(-756)^3$	$(-1404)^3$
7	2	-544825440	2221502112	-2222761824	$(1188)^3$	$(-108)^3$	$(-1404)^3$

Case iv

Replace by 1 by

$$1=\frac{\left(1+i\,12\sqrt{2}\right)\left(1-i\,12\sqrt{2}\right)}{289} \tag{22}$$

Repeating the same procedure as explained in case(iii), the general solutions to (11) are expressed by

$$p=\frac{1}{17}\left[r^2-2s^2-48rs\right]$$

$$q=\frac{1}{17}\left[12r^2-24s^2+2rs\right]$$

Since our interest is on finding integer solutions, we find that p, q *and* δ are integers, for the choices of r *and* s

$r=17R$ *and* $s=17S$

Thus,

$$p=\left[17R^2-34S^2-816RS\right], q=\left[204R^2-408S^2+34RS\right], \delta=289R^2+578S^2 \tag{23}$$

using (23) in (10), we evaluate that

$$\alpha = 221R^2 - 442S^2 - 782RS \;,\; \beta = -187R^2 + 374S^2 - 850RS \;,\gamma = 17R^2 - 34S^2 - 816RS \tag{24}$$

On substituting (24) in (5), (6) and (7), we obtain

$$a = \frac{1}{2}\left[\left(221R^2 - 442S^2 - 782RS\right)^3 + \left(-187R^2 + 374S^2 - 850RS\right)^3 - \left(17R^2 - 34S^2 - 816RS\right)^3\right]$$

$$b = \frac{1}{2}\left[\left(221R^2 - 442S^2 - 782RS\right)^3 + \left(17R^2 - 34S^2 - 816RS\right)^3 - \left(-187R^2 + 374S^2 - 850RS\right)^3\right]$$

$$c = \frac{1}{2}\left[\left(-187R^2 + 374S^2 - 850RS\right)^3 + \left(17R^2 - 34S^2 - 816RS\right)^3 - \left(221R^2 - 442S^2 - 782RS\right)^3\right]$$

Hence, the triple $(a\ ,b,c)$ is integer when S is arbitrary and $R = 2T$

Hence,

$$a = 4\left[\left(442T^2 - 221S^2 - 782ST\right)^3 + \left(-374T^2 + 187S^2 - 850ST\right)^3 - \left(34T^2 - 17S^2 - 816ST\right)^3\right]$$

$$b = 4\left[\left(442T^2 - 221S^2 - 782ST\right)^3 + \left(34T^2 - 17S^2 - 816ST\right)^3 - \left(-374T^2 + 187S^2 - 850ST\right)^3\right]$$

$$c = 4\left[\left(-374T^2 + 187S^2 - 850ST\right)^3 + \left(34T^2 - 17S^2 - 816ST\right)^3 - \left(442T^2 - 221S^2 - 782ST\right)^3\right]$$

Some numerical examples are tabulated below

T	S	a	b	c	$(a+b)$	$(b+c)$	$(c+a)$
1	2	$-2.311845558\times10^{10}$	$-4.145927414\times10^{10}$	4466663776	$(-4012)^3$	$(-3332)^3$	$(-2652)^3$
1	1	− 3 **126534940**	1714067092	− 5794726284`	$(-1122)^3$	$(-1598)^3$	$(-2074)^3$
0	1	−16998980	−69351908	69312604	$(-442)^3$	$(-34)^3$	$(374)^3$
3	2	$-1.451033404\times10^{12}$	1.418388131×10^{12}	$-2.226903797\times10^{12}$	$(-3196)^3$	$(-9316)^3$	$(-15436)^3$
2	0	8703477760	3.55081769×10^{10}	3.548805325×10^{10}	$(3536)^3$	$(272)^3$	$(-2992)^3$

Conclusion

In this communication, we search for the triple $(a\ ,b,c)$ such that the sum of any two of them is a cubical integer. To conclude, one can search for various triples, quadruples, quintuples etc. such that the sum and difference of any two of them is a bi-quadratic integer.

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