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RESEARCH ARTICLE

ON THE CLIFFORD FOURIER TRANSFORMS IN LEBESGUE SPACE SHEAF

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ABSTRACT

In this research paper, we treated sheaves of groups of Hilbert spaces and sheaf of Lebesgue spaces. We also introduced the notion of Fourier transforms and Clifford algebras from which we constructed the Clifford Fourier transform sheaf morphisms.

INTRODUCTION

In [Mallios, 1998; Mallios, 2006] and [Papatriantafillou, 1999] the authors presented sheaves over topological space X which represent a background in this research paper. Sheaves over a topological space X play an important role in Absract mathematics and in some physical applications where we consider the differentiability and integrability as quiet restrictive properties (if not non-natural) and the quest for some algebraic methods seems to be the most desirable. In applied mathematics, the Fourier Transform (FT) has developed into an important tool. It plays a powerful role in solving partial differential equations. The Fourier Transform (FT) avails also a technique in treating signal analysis where the signal from the original domain is transformed to the spectral or frequency domain. The Fourier transform has contribution in the analysis of Lorenz-Lorentz gauge invariance when considering the function as the pertinent generic source function where sources are well-behaved enough in time. All facts in min, we extend the Fourier Transform (FT) in geometric algebra. The Clifford Fourier transform (CFT) was introduced by B. [Jancewicz ,1990], the author proved that CFT plays an important role for electromagnetic field computations in the realm of Clifford's geometric algebra of $\mathbb{R}^3$ where the imaginary complex unit replaces the central unit pseudoscalar of the Clifford algebra. In [Hitzer, 2008], the author expanded the Fourier Transform to multivector valued function distribution in $Cl_{0,n}$ with compact support. [M. Felsberg ,2002] and [Ebling.J, 2005] adapted the same method in Clifford's geometric algebra of $\mathbb{R}^2$ with usual applications in image structure computations and [G.Scheuermann, 2005] suggested Clifford Fourier Transform (CFT) in the study of vector fields in $\mathbb{R}^2$ and $\mathbb{R}^3$ dimensional physical flows. In [De Bie, 2011], the author presented how Fourier Transforms (FTs) in a given space can be generalized to Clifford algebras of that space with the help of an operator form for the complex Fourier Transforms. In the present research paper, we recall the same notions but through sheaf theory so that we come up with a local and global concepts of Clifford Fourier Transforms (CFTs) limiting ourselves in Lebesgue space $L^2(\mathbb{R}_X^2, (\mathbb{R}_X^n, Q_X))$.

1. Preliminaries

Considering X as a fixed topological space (see [Schapira, 1971]). The sheaf of sets over the topological space X is the triplet (S, s, X) such that $s: S \to X$ is a surjective (local) homeomorphism (see [Mallios, 1998], [Mallios, 1998] & [Vassiliou, 2005]).

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For any $x \in X$, we have $s^{-1}(x) = S_x$. In this case, S_x is a fiber of S over x or a stalk of S at x.

We set

$$S = \cup_{x \in X} S_X = \sum_{x \in X} S_X \tag{1.1}$$

If S_x is a Group (Ring, vector space, module, algebra…), then the triplet (S, s, X) is a sheaf of Groups (Rings, vector spaces, modules, algebras…) over X. The triplet $S_X = (S, s, X)$ is a sheaf of $\mathcal{A}-$ modules if the following conditions are satisfied

(i) S is a sheaf of abelian groups,
(ii) The stalk at $x \in ims\ S_x := s^{-1}(\{x\})$ is a left $\mathcal{A}_X-$ module,
(iii) The exterior multiplication $\cdot$: $\mathcal{A} \times_\times S \to S,\ (u, \beta) \to u \cdot \beta \in S_x \subseteq S$, with $a(u) = s(\beta) = x \in X$, is continuous.

We denote a sheaf of sets over X by

$$\mathrm{S_X} \equiv (S, s, X) \tag{1.2}$$

Let $\mathcal{A}_\mathrm{X} \equiv (\mathcal{A}, a, X)$ be the sheaf of $\mathbb{K}_X$-algebras (or simply, the $\mathbb{K}_X$-algebra sheaf), which is preferably unital and commutative, with $\mathbb{C}_X$ the sheaf of complex numbers over X. We denote by $\mathcal{A}_X^+$ and $\mathcal{A}_X^-$ the sub sheaves of $\mathcal{A}_X$ formed by positive elements and negative elements, respectively. In this case, we have

$$\mathcal{A}_X^+ \cap \mathcal{A}_X^- = \{0\}_X \quad \text{and} \quad \mathcal{A}_\mathrm{X}^+ \cup \mathcal{A}_\mathrm{X}^- = \mathcal{A}_\mathrm{X} \tag{1.3}$$

The sheaf $\mathrm{S_X} \equiv (S, s, X)$ is called vector sheaf (or free module) of rank n iff, for any open $x \in X$, we have the following relation

$$S_x \cong \mathcal{A}_x^n \equiv (\mathcal{A}^n)_x \cong S|_U \cong \bigoplus^n (\mathcal{A}|_U) \coloneqq (\mathcal{A}|_U)^n \tag{1.4}$$

where $\mathcal{A}_X$ is the sheaf of $\mathbb{K}_X$-algebras , with $\mathbb{K}_X = (\mathbb{R}_X\ or\ \mathbb{C}_X)$ and $\mathcal{A}_X$ is a sheaf of unital $\mathbb{K}$-algebras over X, in other words $(\mathcal{A}|_U)^n$ denotes the n-terms direct sum of the sheaf of $\mathbb{C}$-algebras $\mathcal{A}$ restricted to U, for some $n \in \mathbb{N}$. Here $(\mathcal{A}|_U)^n$ is the local sectional frame or equivalently local gauge of states of S associated via the open covering $\mathcal{U} = \{U_i\}$ of X. For $n = 1$, the corresponding vector sheaf is termed as a line sheaf of states, that is locally for any point $x \in X$ there exists an open set U of X such that $S(U) \equiv \ S|_U \cong \mathcal{A}|_U$.

We observe that $\mathbb{R}_X \equiv (\mathbb{R}, r, X)$ and $\mathbb{C}_X \equiv (\mathbb{C}, c, X)$ are respectively the sheaf of real and complex numbers. In this regards, for any $x \in U$ open in X, we have

$$S_x = \underset{\overleftarrow{x \epsilon U}}{lim} S_U \equiv \underset{\overleftarrow{x \epsilon X}}{lim} S_X \tag{1.5}$$

where $\underset{\overleftarrow{\mathrm{x \epsilon X}}}{\mathrm{lim}}$ represents the inductive limit.

Letting $\mathcal{A}lg_X$ be the category of sheaves of unital $\mathbb{K}$-algebras over X and $DVect_X\ (or\ Mod_X)$ the category of (differential) vector sheaves (or the category of sheaves of (differential) free $\mathcal{A}_X$-modules) over X (see [Herich, 1973] & [Kashiwara, 2006]). Consider the following functor

$$F_X^{diff}: Alg_X \ \to \ DMod_X \tag{1.6}$$

such that, for any $\mathcal{A}_X \in\ Alg_X$, there exists $\Omega_X \in\ DMod_X$ satisfying

$$F_X^{diff}(\mathcal{A}_X) = \Omega_X \tag{1.7}$$

so that, for any $x \in U$ open in X, the Leibniz (product) rule

$$F_U^{diff}\big|_{\mathcal{A}_U}(a.b) = a.F_U^{diff}\big|_{\mathcal{A}_U}(b) + b.F_U^{diff}\big|_{\mathcal{A}_U}(a) \tag{1.8}$$

is satisfied , for any (local) sections $a, b \in \mathcal{A}_U$ and $F_U^{diff}\big|_{\mathcal{A}_U}: \mathcal{A}_U \to \Omega_U$ is continuous and $\mathbb{K}_U$-linear. In the previous expression, we have identified a sheaf with the sheaf of germs of its sections

The symbol " $\big|$ " designs the restriction of the functor to the object and F_X^{diff} represents the differential functor and $F_X^{diff}\big|_{\mathcal{A}_X}$ is the differential morphism, all over X.

For simplicity, we express the differential morphism over X as

$$F_X^{diff}\big|_{\mathcal{A}_X} = \ d_X \tag{1.9}$$

and define the triplet

$$(\mathcal{A}_X, d_X, \Omega_X) \in Alg \ \times_X \ F^d \ \times_X \ DMod \tag{1.10}$$

as the differential triad over a $\mathbb{C}$- algebraized space $(\mathcal{A}, X)$ (see [M. H. Papatriantafillou, 2000]).

In this regards, one writes

$$dT_X = (\mathcal{A}_X, d_X, \Omega_X) \tag{1.11}$$

The following functor

$$F_X^{int}: DMod_X \to Alg_X \tag{1.12}$$

is such that, for any $\Omega_X \in DMod_X$, there exists $\mathcal{A}_X \in Alg_X$ satisfying

$$F_X^{int}(\Omega_X) = \mathcal{A}_X \tag{1.13}$$

and for any $x \in U$ open in X and any differential form f in Ω_U , we have

$$F_U^{int}\big|_{\Omega_U}(f) = F + Constant \in \mathcal{A}_U \subset \mathcal{A}_X \tag{1.14}$$

with $F \in \mathcal{A}_U$ and Constant $\in \mathbb{K}_U$. Consequently, one writes

$$F_U^{int}\big|_{\Omega_U}(f) = \int_{\Omega_U} f d_U \eta \tag{1.15}$$

where η is a measure defined in Ω_U and $\int_{\Omega_U}: \Omega_U \to \mathcal{A}_U$ is continuous and satisfies the following relation, for any $x \in U$, with U open in X

$$\int_{\Omega_U} = d_U^{-1} \equiv (d_U)^{-1} \tag{1.16}$$

and the triplet

$$\left(\Omega_X, \int_{\Omega_X}, \mathcal{A}_X\right) \in DMod \times_X F^{int} \ \times_X Alg \tag{1.17}$$

is defined as the integral triad over a $\mathbb{C}$- algebraized space $(\mathcal{A}, X)$ denoted as

$$intT_X = \left(\Omega_X, \int_{\Omega_X}, \mathcal{A}_X\right) \tag{1.18}$$

Moreover, for any $x \in U$ open in X, we have

$$dT_x = \varinjlim_{x \epsilon U} dT_U = \varinjlim_{x \epsilon X} dT_X \tag{1.19}$$

and

$$intT_x = \varinjlim_{x \epsilon U} intT_X = \varinjlim_{x \epsilon X} intT_X \tag{1.20}$$

where $\varinjlim_{x \epsilon X}$ represents the inductive limit.

Through quadratic differential triad spaces, we construct the Clifford differential triad algebras. From the Clifford differential triad morphisms , we realize the category of Clifford differential triad algebras and other important functions. Letting (E_X, q_{E_X}) be a quadratic space, the Clifford algebra of (E_X, q_{E_X}) or simply, of E, is regarded as a pair (C_X , c_X) formed by an $\mathcal{A}_X$–algebra C_X and an $\mathcal{A}_X$ –linear map $c_X: E_X \to C_X$ such that , for any open $U \subseteq X$, we have

$$c_U(s)^2 = q_{E_U}(s) \,.\, 1_{C_U} \tag{1.21}$$

with $s \in E_U \equiv E(U)$ and $q_{E_U} : E_U \to \mathcal{A}_U$, where 1_{C_U} designs the unity in C_U. In presheaves notation, we have

$$c_V^U \ o \ c_U = c_V \ o \ e_V^U \ , \ a_V^U \ o \ q_{E_U} = q_{E_V} \ o \ e_V^U$$

with $V \subseteq U \subseteq X$, open,

For any $\mathcal{A}_X$-algebra and all $\mathcal{A}_X$–linear map f_X: $E_X \to F_X$ and any open $U \subseteq X$, we have

$$f_X(s)^2 = q_{E_U}(s) \,.\, 1_{F_U} \tag{1.22}$$

where $s \in E_U$. Also, there exists a unique morphism of $\mathcal{A}_X$–algebras $\tau_X : C_X \to F_X$ such that the following diagram commutes

$$\begin{tikzcd} E_X \arrow[r, "f_X"] \arrow[d, "c_X"'] & F_X \\ C_X \arrow[ur, "\tau_X"'] & \end{tikzcd}$$

In other words, one obtains

$$f_V^U \ o\ \tau_U = \tau_V \ o\ c_V^U\ ,\ a_V^U \ o\ q_{E_U} = q_{E_V} \ o\ e_V^U,$$

with $V \subseteq U \subseteq X$, open.

We denote the Clifford $\mathcal{A}_X$–algebra (C_X, c_X) by

$$C_X \equiv C_X(\mathrm{E}_X, q_{E_X}) \equiv (C_X(\mathrm{E}_X, q_{E_X}), c_X). \tag{1.23}$$

2. Fourier transform in $L^2\left(\mathbb{R}_X^2, (\mathbb{R}_X^n, \boldsymbol{Q}_X)\right)$

Let X be a fixed topological space and $G_X \equiv (G, \sigma, X)$ be a sheaf (of sets) over X ; i.e., $\sigma : G \to X$ is a (local) homeomorphism. We say that G_X is a sheaf of groups (see [Mallios, 2006], [Mallios, 2002] & [Vassiliou, 2005]) (or shortly a group sheaf) over X if, for any $x \in X$, the fiber

$$G_X = \sigma^{-1}(\{x\}) \equiv \sigma^{-1}(x) \subseteq G \tag{2.1}$$

is a classical group.

Consider a subsheaf $G\nabla G$ of $G \times G$ defined by

$$G \nabla G = \{ (g, g') \in G \times G : \sigma(g) = \sigma(g')\ x \in X\} \tag{2.2}$$

We say that $G_X \equiv (G, \sigma, X)$ is a topological group sheaf if it is a group sheaf over X and the map

$$G\nabla G \xrightarrow{p} G \tag{2.3}$$

is continuous and for any $x \in U$ open in X , we have the following inductive limit

$$G_x = \underset{x \in U}{\underrightarrow{\lim}}\, G_U$$

Let $\mathbb{K}_X = (\mathbb{R}_X\ or\ \mathbb{C}_X) \equiv (\mathbb{K}, \blacksquare, X)$ be a sheaf of fields (or shortly a field sheaf) over X . We say that $V_X \equiv (V, \varepsilon, X)$ is a sheaf of $\mathbb{K}_X$–vector spaces (or shortly a $\mathbb{K}_X$–vector space sheaf) over X if, for any $x \in X$, the fiber

$$V_X = \varepsilon^{-1}(\{x\}) \equiv \varepsilon^{-1}(x) \tag{2.4}$$

is a classical vector space.

Consider the following subsheaves $V\nabla V$ of $V \times V$ and $\mathbb{K}\nabla V$ of $\mathbb{K} \times V$ defined by:

$$V\nabla V = \{(v, v') \in V \times V : \varepsilon(v) = \varepsilon(v') = x \in X\} \tag{2.5}$$

and

$$\mathbb{K}\nabla V = \{(k, v) \in \mathbb{K} \times V : \blacksquare(k) = \varepsilon(v) = x \in X\} \tag{2.6}$$

We observe that $V_X \equiv (V, \varepsilon, X)$ is a topological $\mathbb{K}_X$– vector space sheaf over X , if V_X is a classical topological $\mathbb{K}_X$–vector space and the maps

$$V\nabla V \xrightarrow{+} V \text{ and } \mathbb{K}\nabla V \xrightarrow{*} V \tag{2.7}$$

are continuous. Let G_X and V_X be respectively a topological group sheaf and a topological vector sheaf. Letting $GL(V)$ be a linear group sheaf of V and we consider two continuous linear morphism

$$G \xrightarrow{\varphi} GL(V)\ ,\quad V\nabla G \to V \tag{2.8}$$

given respectively, for any $x \in X$, by

$$g_x \to \varphi(g_x) = \varphi(g_x)\ ,\ (v, g)_x \equiv (v_x, g_x) \to \varphi(g_x)\,(v_x) \tag{2.9}$$

with $g_x \in G_x \subseteq G$, and $v_x \in V_x \subseteq V$, then the pair

$$(\varphi, V)_X \equiv (\varphi_X, V_X) \tag{2.10}$$

is a continuous linear representation from G to V. We notice that if $V_X = (V, <\cdots>)_X \equiv (V_X, <\cdots>_X)$ is an Hilbert space sheaf over X ; i.e., the sheaf morphism norm $\|\ \|_X : V_X \to \mathbb{R}_X^+$ is complete (in other words, all Cauchy sequences in V_X converge) and the following scalar product $<\cdots>_X : V_X \to \mathbb{R}_X^+$ is such that, for any $x \in X$, we have

$$\|v_x\|_x^2 = < v_x, v_x >_x \tag{2.11}$$

$$v_x \in V_x \subseteq V$$

and V_x is finite-dimensional, then the representation $(\varphi, V)_X = (\varphi_X, V_X)$ is said to be finite, and the dimension of V_x is called the degree of the representation. Letting by W_X a subsheaf of V_x then, it is said to be invariant by φ_X if, for any $x \in X$ and $g_x \in G_x \subseteq G$,

$$\varphi(g_x)(W_x) \equiv \varphi_x(g_x)(W_x) \subset W_x \subseteq V \quad (2.12)$$

The representation (φ_X, V_X) is regarded as irreducible if W_X and $\{0_X\}$ are only the subsheaves of V_X that are invariant by φ_X. Assuming that (φ_{1X}, V_{1X}) and (φ_{2X}, V_{2X}) are two linear representations of the same group sheaf G_X. They are equivalent if there exists a $\mathbb{K}_X$-isomorphism γ_X: $V_{1X} \rightarrow V_{2X}$ such that, for any $x \in X$ and $g_x \in G_x \subseteq G$,

$$\gamma_x o\ \varphi_{1x}(g_x) = \varphi_{2x}(g_x)\ o\ \gamma_x \quad (2.13)$$

In this regard, V_X is a $\mathbb{C}_X$–vector space sheaf equipped with a Hermitian form

$$< , >_X : V_X \nabla V_X \rightarrow V_X \quad (2.14)$$

i.e., a $\mathbb{C}_X$–bilinear form considered as follows. The representation (φ_X, V_X) is said to be unitary with respect to $< , >_X$ if, for any $x \in X$,

$$< \varphi(g_x)(v_x), \varphi(g_x)(v'_x) >_X = < v_x , v'_x >_X \quad (2.15)$$

with any v_x , $v'_x \in V_x \subseteq V_X$ and $g_x \in G_x \subseteq G$. We now restrict ourselves to locally compact unimodular group sheaves, i.e, we consider a measure that is invariant with respect to both left and right translations, called a Haar measure. By letting G_X be a locally compact unimodular group sheaf and denoting $\upsilon_X: \sigma(G_X) \rightarrow \mathbb{R}_X^+$ a Haar measure, then, for any $f_X \in L^2(G_X, \mathbb{C}_X)$ and $h \in G_X$, we have (see [B. Jancewicz, 1990])

$$\int_{G_X} f_x(g_x)\, d\upsilon_x(g_x) = \int_{G_X} f_x(g_x h_x)\, d\upsilon_x(g_x) = \int_{G_X} f_x(h_x g_x)\, d\upsilon_x(g_x) \quad (2.16)$$

For any $x \in X$ and $g_x,\ h_x \in G_x \subseteq G_X$.

We see that locally compact abelian group sheaves and compact group sheaves are unimodular sheaves. Let $f_X \in L^2(G_X, \mathbb{C}_X)$, where G_X is a locally compact unimodular sheaf over X and υ_X be the Haar measure. The Fourier transform of f_X as is thus the map $\hat{f}_X$ given, for any $x \in X$, by

$$\hat{f}_x(\varphi_x) = \int_{G_X} f_x(g_x)\varphi_x(g_x^{-1})\, d\upsilon_x(g_x) \quad (2.17)$$

Then, we say that $\hat{f}_X \in L^2(\hat{G}_X, \hat{\mathbb{C}}_X)$ and $\hat{f}_x(\varphi_x)$ is a Hilbert-Schmidt operator over the space sheaf of the representation φ_x.

Theorem 2.1. Let $f_X \in L^2(G_X, \mathbb{C}_X)$, where G_X is a locally compact unimodular sheaf over X and $\hat{f}_X \in L^2(\hat{G}_X, \hat{\mathbb{C}}_X)$. There is a measure over $\hat{G}_X$ denoted by $\hat{\upsilon}_X$ such that $f_X \rightarrow \hat{f}_X$ is an isometry. The following inverse formula holds

$$f_x(g_x) = \int_{\hat{G}_X} Trace\left(\hat{f}_x(\varphi_x)\varphi_x(g_x)\right) d\hat{\upsilon}_x(\varphi_x) \quad (2.18)$$

where f_X is the Fourier inverse transform of $\hat{f}_X$. Let f_X be a real or complex sheaf valued function defined on $G_X = \mathbb{R}_X^2 \equiv (\mathbb{R}_X)^2$. Then, its Fourier transform is given, for any $x \in X$, by

$$\hat{f}_X(r_x, r'_x) = \int_{\mathbb{R}_X^2} \hat{f}_x(\alpha_x, \beta_x) e^{-i(r_x\alpha_x + r'_x\beta_x)} d\alpha_x d\beta_x \quad (2.19)$$

Also, if we identify $\mathbb{C}_X$ with $(\mathbb{R}_X^2, \|\ \|_{2X})$, then we have a special group sheaf $SO(2)_X$, called the 2–special orthogonal group sheaf over X, such that the action of $SO(2)_X$ on $\mathbb{C}_X$ (given by the complex multiplication) corresponds to the action of S_X^1 on $(\mathbb{R}_X^2, \|\ \|_{2X})$. Assuming that G_X is a group sheaf. We say that G_X is a Lie group sheaf over X if, the group morphism sheaves $G \nabla G \overset{\phi}{\rightarrow} G$ and $G \overset{\psi}{\rightarrow} G$, given, for any $x \in X$, respectively by

$$\phi_x(g_x, g'_x) = g_x g'_x \text{ and } \psi_x(g_x) = g_x^{-1} \quad (2.20)$$

are derivable. Assuming that G_X is a Lie group sheaf over X, then $\hat{G}_X$ is the Pontryagin dual sheaf of G_X if it is the sheaf of equivalence classes of unitary irreducible representations of G_X. From the above concept, one says that the Pontryagin dual of $\mathbb{R}_X^n$ is $\mathbb{R}_X^n$; i.e., $\widehat{\mathbb{R}}_X^n = \mathbb{R}_X^n$. The Pontryagin dual of $SO(2)_X$ is $\mathbb{Z}_X$. By letting G_X to be a Lie group sheaf over X and S_X^1 a 1–spherical sheaf over X. The character of G_X is a continuous group morphism from G_X to S_X^1. If $G_X = SO(2)_X$, then the characters of G_X are the group morphisms from $SO(2)_X$ to S_X^1 such that, for any $x \in U$ open in X, we have $\theta_x \in SO(2)_X \rightarrow e^{in\theta_x} \in S_X^1$, $n \in \mathbb{Z}$.

If $G_X = \mathbb{R}_X^m$, then the characters of G_X are the group morphisms from $\mathbb{R}_X^m$ to S_X^1 given, for any $x \in U$ open in X, by

$$(\theta_{1x}, \dots, \theta_{mx}) \in \mathbb{R}_X^n \rightarrow e^{i(n_1\theta_{1x} + \dots + n_m\theta_{mx})} \in S_X^1$$

with $n_1, \dots, n_m$ are real numbers.

For $Q_X: V_X \rightarrow \mathbb{K}_X$ considered as a $\mathbb{K}$–quadratic form on V_X defined for any $x \in U$ open in X as follows

$$v_x \rightarrow Q(v_x) \equiv Q_x(v_x), \quad v_x \in V_x \subseteq V_X$$
$$k_x v_x \rightarrow Q(k_x v_x) \equiv Q_x(k_x v_x) = k_x^2\, Q_x(v_x), \; k_x \in \mathbb{K}_x \subseteq \mathbb{K}_X, v_x \in V_x \subseteq V_X$$
$$(v_x, v'_x) \rightarrow B(v_x, v'_x) \equiv B_x(v_x, v'_x) = Q_x(v_x - v'_x) - Q_x(v_x) - Q_x(v'_x)$$
$$v_x, v'_x \in V_x \subseteq V_X, \text{ where } B_x \text{ is bilinear.}$$

Then the symmetric $\mathbb{K}_X$–bilinear form B_x is regarded as a polar $\mathbb{K}_X$–bilinear form, with

$$B_X = \sum_{x \in X} B_x \tag{2.21}$$

We notice that for any $v_x \in V_x \subseteq V_X$, one writes

$$Q_x(v_x) = B_x(v_x, v_x) \tag{2.22}$$

For $Q_x(v_x) = 0_x$, we have a $\mathbb{K}_X$–quadric. Referring to the above consideration, a slightly more general concept is to concentrate in the first place on inner product. Let given the symmetric $\mathbb{K}_X$–bilinear inner product $< | >_x : V_X \nabla V_X \rightarrow \mathbb{K}_X$, defined, for any $x \in U$, by

$$< | >_x(v_x, v'_x) = < v_x | v'_x >_x = < v'_x | v_x >_x \in \mathbb{K}_x \subseteq \mathbb{K}_X \tag{2.23}$$

We observe that there is need to introduce dual space sheaves for polar elements, i.e., hyperplanes. For this reason, we have to assume that the characteristic of $\mathbb{K}_X$ is not equal to 2 . The major concern is to find out about the kind of algebra sheaves which arise from adding this particular structure to and algebra sheaf having a product m . Such structure

$$(V_X, m_X, Q_X) \tag{2.24}$$

would , e.g., be an operator algebra sheaf where we have employed a non-canonical quantization. However, it is more suitable to find out if the quadratic form can imply a product on V_X . In this case, the product map m_X is a consequence of the quadratic form Q_X itself, since Clifford algebra sheaves enjoy the goodness of this type of quadratic form.

3. Main result: Clifford Fourier transform in $L^2\left(\mathbb{R}_X^2, (\mathbb{R}_X^n, Q_X)\right)$

From its natural construction, based on quadratic form having a symmetric polar bilinear form B_{px} , it is clear that we can expect Clifford algebra sheaves to be related to orthogonal group sheaves. Clifford algebra sheaves should be interpreted as a linearization of a quadratic form. It was introduced for the first time by Dirac who used this approach to postulate his famous equation. Furthermore, we can learn from the polarization process that this type of algebra is related to anti-commutative relations

$$Q_x(v_x) = \sum_i v^i v^j \left(e_i e_j\right)_x, \; 2B_{px} = v^i v'^i \left(e_i e_j + e_j e_i\right)_x \tag{3.1}$$

for any $x \in U$ open in X , which leads necessarily to

$$\left(e_i e_j + e_j e_i\right)_x = 2B_{pij,x} \tag{3.2}$$

Anticommutative algebra sheaves are referred to as (canonical) anticommutation algebra sheaves.

Let V_X be a $\mathbb{K}_X$-vector space sheaf over X . The $\mathbb{K}_X$–bilinear form $< | >_x : V_X \nabla V_X \rightarrow \mathbb{K}_X$ is antisymmetric if, for any $x \in U$ open in X , we have

$$< v_x | v'_x >_x = -< v'_x | v_x >_x$$

A $\mathbb{K}_X$–linear space sheaf equipped with an antisymmetric $\mathbb{K}_X$–bilinear inner product is regarded as Weyl space sheaf. The antisymmetric implies directly that all vectors are null or synonymously isotrop

$$< v_x | v_x >_x = 0_x, \; x \in U \text{ open in } X \tag{3.3}$$

It is possible to define a $\mathbb{K}_X$–algebra as follows

$$(V_X, m_X, < | >_x) \equiv (V, m, < | >)_X \tag{3.4}$$

Moreover, we are interested in such products which are derived from $\mathbb{K}_X$–bilinear form. From polarization technique, we find a (canonical) commutator relation $\mathbb{K}_X$–algebra sheaf

$$\left(e_i e_j - e_j e_i\right)_x = 2B_{ij,x} \tag{3.5}$$

Considering the action of group morphism sheaves from $\mathbb{R}_X^2$ to $SO(2)_X$ on $(\mathbb{R}_X^2, \| \ \|_{2X})$, for any $x \in U$ open in X and $f \in L^2\left(\mathbb{R}_X^2, (\mathbb{R}_X^2, \| \ \|_{2X})\right)$, we define the Fourier transform F as in (2.19).

Consider the Clifford algebra sheaf configuration. Let us embed $(\mathbb{R}^2_X, \|\ \|_{2X}) \equiv (\mathbb{R}^2, \|\ \|_2)_X$ into $\mathbb{R}_{2,0X} \equiv (\mathbb{R}_{2,0})_X$ so that f_X can be regarded as $(\mathbb{R}^1_{2,0})_X$–valued function

$$f_x(g_x, g'_x) = f_{1x}(g_x, g'_x)e_{1x} + f_{2x}(g_x, g'_x)e_{2x} \tag{3.6}$$

for any $x \in U$ open in X, where

$$e^2_{1x} = e^2_{2x} = 1_x \equiv 1 \text{ and } e_{1x}e_{2x} + e_{2x}e_{1x} = 0_x \equiv 0 \tag{3.7}$$

With above conditions, (2.19) becomes

$$\hat{f}_x(r_x, r'_x) = \int_{\mathbb{R}^2_X}\left[cos\left(\left(\frac{\alpha_x r_x+\beta_x r'_x}{2}\right)\right)1_x + sin\left(\left(\frac{\alpha_x r_x+\beta_x r'_x}{2}\right)\right)e_{1x}e_{2x}\right](f_{1x}(r_x, r'_x)e_{1x} + f_{2x}(r_x, r'_x)e_{2x})\left[cos\left(-\left(\frac{\alpha_x r_x+\beta_x r'_x}{2}\right)\right)1_x + sin\left(-\left(\frac{\alpha_x r_x+\beta_x r'_x}{2}\right)\right)e_{1x}e_{2x}\right]d\alpha_x d\beta_x \tag{3.8}$$

Let Q_X be a non-degeneracy $\mathbb{K}_X$–quadratic form and B_X the associated $\mathbb{K}_X$-bilinear form defined on V_X (see [Ebling, 2005], [Ebling, 2003] & [H. De Bie & Y. Xu, 2011]). Denote by $\mathcal{C}(V_X, Q_X) \equiv \mathcal{C}(V_X)$ is the Clifford $\mathbb{K}_X$–algebra sheaf on G_X. We say that a sheaf G_X is a Clifford group sheaf if, for any $x \in U$ open in X and $g_x \in \mathcal{C}(V_x) \equiv \mathcal{C}(V_X)$, we have (see [E. Hitzer, B. Mawardi, 2008] & [G. Sommer, 2001])

$$G_x = \{g_x \in \mathcal{C}(V_x): \phi_{g_x}(v_x) = g_x r_x g_x^{-1} \in V_x\} \tag{3.9}$$

where $r_x \in V_x$ and $\phi_{g_x} \in Aut_{\mathbb{K}_x}(V_x)$.

We notice that G_X is a multiplicative group sheaf over X of $\mathcal{C}(V_X)$. The $\mathbb{K}_X$–automorphism $\phi_{g_x}: V_X \to V_X$ is a $\mathbb{K}_X$–isometry; i.e., an orthogonal transformation of $O(V_X)$. In other words, $\phi_{g_x}: V_X \to V_X$ is a $\mathbb{K}_X$–linear bijective transformation, if for any $x \in U$ open in X, we have

$$Q\left(\phi_{g_x}(v_x)\right) \equiv Q_x\left(\phi_{g_x}(v_x)\right) = Q_x(v_x) \tag{3.10}$$

For any $g_x \in G_x \subseteq G_X$ and $v_x \in V_x \subseteq V_X$. Let G_X be a Clifford group sheaf over X. We define a group spin denoted $Spin(n)_X$, as follows

$$Spin(n)_X = \{\textstyle\prod_{i=1}^{2k}(a_i)_x, (a_i)_x \in \mathbb{R}^1_{n,0x}, \|a_i\|_x = 1_x\} \tag{3.11}$$

Or equivalently

$$Spin(n)_X = \{g_x \in \mathbb{R}_{n.0x}, \alpha(g_x) = g_x,\ g_x g_x^+ = 1_x, g_x v g_x^{-1} \in \mathbb{R}^1_{n,0x}, \forall v \in \mathbb{R}^1_{n,0x}\} \tag{3.12}$$

It is well known that $Spin(n)_X$ is a connected compact Lie group that universally covers $SO(n)_X$ $(n \gg 3)$. Let $\varphi_{r_X,r'_X} \equiv (\varphi_{r,r'})_X$ be a sheaf morphism from $\mathbb{R}^2_X$ to $Spin(2)_X$. Then, we can write expression (219), for any $x \in U$ open in X, in the following form (see [G. Sommer, 2001])

$$\hat{f}_x(r_x, r'_x) = \int_{\mathbb{R}^2_X}(f_{1x}(r_x, r'_x)e_{1x} + f_{2x}(r_x, r'_x)e_{2x}) \perp_x \varphi_{r_X,r'_X}(-\alpha_x, -\beta_x)d\alpha_x d\beta_x \tag{3.13}$$

Where φ_{r_X,r'_X} sends (α_x, β_x) to $exp_x\left[\left(\frac{\alpha_x r_x+\beta_x r'_x}{2}\right)\right]e_{1x}e_{2x}$ and $\perp_x$ denotes the action $v_x \perp_x s_x = s_x^{-1}v_x s$ of $Spin(2)_X$ on $\mathbb{R}^1_{2,0x} \equiv (\mathbb{R}^1_{2,0})_x$.

The group sheaf morphisms $(\varphi_{r,r'})_X$ followed by the action on $(\mathbb{R}^1_{2,0})_x$ correspond to the action of group sheaf morphisms from $\mathbb{R}^2_X$ to $SO(2)_X$ on $(\mathbb{R}^2, \|\ \|_2)_X$. The Fourier transform of a real valued function is then defined by embedding $\mathbb{R}_X$ to $\mathbb{R}^2_X$. Let $f_X \in L^2(\mathbb{R}^2_X, (\mathbb{R}^n_X, Q_X))$, where $Q_X: \mathbb{R}^2_X \to \mathbb{R}^2_X$ is a positive definite quadratic form. Let us associate the Fourier transform of f_X with the action of the following group morphisms on the values of f_X, depending on the parity of n.

If n is even, then we consider the morphisms

$$\varphi_X: \mathbb{R}^2_X \to SO(Q)_X \tag{3.14}$$

where $SO(Q)_X$ is the special orthogonal group sheaf over X relative to the quadratic space sheaf $\mathbb{R}^2_X$.
If n is odd, then we embed $(\mathbb{R}^n_X, Q_X)$ into $(\mathbb{R}^{n+1}_X, Q_X \oplus 1_X)$ and we consider the morphism

$$\varphi_X: \mathbb{R}^2 \to SO(Q_X \oplus 1_X) \equiv SO(Q \oplus 1)_X \tag{3.15}$$

The Fourier transform in this regard depends on the positive definite quadratic form $\mathbb{R}^p_X$ (p denotes n if n is even and $n+1$ if n is odd). For this reason, $SO(2)_X$ becomes $SO(p)_X$ and the group sheaf morphisms from $\mathbb{R}^2_X$ to $SO(Q)_X$ become group morphisms from $\mathbb{R}^2_X$ to $SO(p)_X$. For the case of $\mathbb{R}^2_X$ –valued functions, the Fourier transform can be written in the Clifford algebra sheaf configuration where we consider the action of $Spin(p)$ on $\left(\mathbb{R}^1_{p,0}\right)_X$ which corresponds to the action of $SO(p)_X$ on $\mathbb{R}^p_X$ .Denoting by φ_X a group sheaf morphism from $\mathbb{R}^2_X$ to $SO(p)_X$, then we define the Clifford-Fourier transform of $f_X \in L^2\left(\mathbb{R}^2_X, (\mathbb{R}^n_X, Q_X)\right)$ as follows

$$\hat{f}_x(\varphi_x) = \int_{\mathbb{R}^2_X} \varphi_x(\alpha_x, \beta_x) f_x(\alpha_x, \beta_x) \varphi_x(-\alpha_x, -\beta_x) d\alpha_x d\beta_x$$
$$\hat{f}_x(\varphi_x) = \int_{\mathbb{R}^2_X} f_x(\alpha_x, \beta_x) \perp_x \varphi_x(-\alpha_x, -\beta_x) d\alpha_x d\beta_x \tag{3.16}$$

where we have

(if p is even): $f_x(\alpha_x, \beta_x) = f_{1x}(\alpha_x, \beta_x) e_{1x} + \cdots + f_{nx}(\alpha_x, \beta_x) e_{nx}$ (3.17)

(if p is odd) : $f_x(\alpha_x, \beta_x) = f_{1x}(\alpha_x, \beta_x) e_{1x} + \cdots + f_{nx}(\alpha_x, \beta_x) e_{nx} + O e_{n+1,x}$ (3.18)

with $e^2_{i,x} = 1_x$ and $e_{i,x} e_{j,x} = -e_{j,x} e_{i,x}$

We observe that

(1) $f_X\colon \mathbb{R}^2_X \to Spin(3)_X$ is such that, for any $x \in X$,

$$(\alpha_x, \beta_x) \to e^{1/2(u_x\alpha_x + v_x\beta_x)A_x} \tag{3.19}$$

Where $A_x \in \left(S^2_{3,0}\right)_x$ and $u_x, v_x \in \mathbb{R}_x$

(2) $\tilde{\phi}_X\colon \mathbb{R}^2_X \to Spin(4)_X$ is such that, for any $x \in X$, are the sheaf morphisms that send (α_x, β_x) to:

$$e^{1/8[\alpha_x(u_x+w_x)+\beta_x(v_x+t_x)][A_x+B_x+I_x(A_x-B_x)]}\ e^{1/8[\alpha_x(u_x-w_x)+\beta_x(v_x-t_x)][A_x-B_x+I_x(A_x+B_x)]} \tag{3.20}$$

With $u_x, v_x, w_x, t_x \in \mathbb{R}_x$ and $A_x, B_x \in \left(S^2_{3,0}\right)_x$.

Let $f_X \in L^2\left(\mathbb{R}^2_X, (\mathbb{R}^3_X, Q_X)\right)$, resp. $f_X \in L^2\left(\mathbb{R}^2_X, (\mathbb{R}^4_X, Q_X)\right)$ and denote by f_X the embedding of f_X into the Clifford algebra $Cl(\mathbb{R}^4_X, Q_X \oplus 1_X)$, resp. $Cl(\mathbb{R}^4_X, Q_X)$. For any $x \in U$ open in X, the Clifford-Fourier Transform of f_X is given by

$$\hat{f}_x(u_x, v_x, w_x, t_x, D_x) = \int_{\mathbb{R}^2_X} f_x(\alpha_x, \beta_x) \perp_x \tilde{\phi}_{u_x,v_x,w_x,t_x,D_x}(-\alpha_x, -\beta_x) d\alpha_x d\beta_x$$
$$\hat{f}_x(u_x, v_x, w_x, t_x, D_x) = \int_{\mathbb{R}^2_X} e^{1/2[\alpha_x(u_x+w_x)+\beta_x(v_x+t_x)D_x]} e^{1/2[\alpha_x(u_x-w_x)+\beta_x(v_x-t_x)I_xD_x]}$$
$$f_x(\alpha_x, \beta_x) e^{-1/2[\alpha_x(u_x+w_x)+\beta_x(v_x+t_x)D_x]} e^{-1/2[\alpha_x(u_x-w_x)+\beta_x(v_x-t_x)I_xD_x]}\, d\alpha_x d\beta_x \tag{3.21}$$

If we decompose f_X as the sum $(f_\parallel)_X + (f_\perp)_X$ with respect to the plane generated by the bisector D_X, we get for any $x \in U$ open in X

$$\hat{f}_x(u_x, v_x, w_x, t_x, D_x) = \int_{\mathbb{R}^2_X} (f_\parallel)_x(\alpha_x, \beta_x) e^{[-\alpha_x(u_x+w_x)+\beta_x(v_x+t_x)D_x]} d\alpha_x d\beta_x$$
$$+ \int_{\mathbb{R}^2_X} (f_\perp)_x(\alpha_x, \beta_x) e^{[-\alpha_x(u_x-w_x)+\beta_x(v_x-t_x)I_xD_x]} d\alpha_x d\beta_x \tag{3.22}$$

The plane generated by $I_x D_x$ represents the orthogonal of the plane generated by D_x in $\mathbb{R}^4_x \subseteq \mathbb{R}^4_X$. Notice that the Clifford-Fourier Transform of g_X is left-invertible. Its inverse is the map $\breve{g}_X$ given by

$$\breve{g}_x(a_x, b_x) = \int_{\mathbb{R}^4_X \times \left(S^2_{4,0}\right)_x} g_x(u_x, v_x, w_x, t_x, D_x) \perp_x \tilde{\phi}_{(u_x,v_x,w_x,t_x,D_x)}(a_x, b_x) du_x dv_x dw_x dt_x dv_x \tag{3.23}$$

Where v_x is a unit measure on $\left(S^2_{4,0}\right)_x \subseteq \left(S^2_{4,0}\right)_X$.

Also, for any $x \in V \subseteq U$ open in X, we have the following information

$$\begin{array}{ccc} \left(S^2_{4,0}\right)_U & \overset{s^U_V}{\to} & \left(S^2_{4,0}\right)_V \\ \varinjlim_{x\epsilon U} \downarrow & & \downarrow \varinjlim_{x\epsilon U} \\ \left(S^2_{4,0}\right)_x & = & \left(S^2_{4,0}\right)_x \end{array}$$

In other words, we have the following inductive limit

$$\varinjlim_{x\epsilon U} \equiv \varinjlim_{x\epsilon V}\ o\ s^U_V$$

Where s^U_V represents the restriction map and $\varinjlim_{x\epsilon U}\left(or\ \varinjlim_{x\epsilon V}\right)$ is called the inductive limit morphism.

4. Conclusion

The Fourier transform plays an important role in some transformations in mathematics. We showed how sheaves over topological space can be illustrated in abstract mathematics. We have constructed the Clifford differential and integral triad algebras. We have then combined the notion of Fourier transform and Clifford's geometric algebra to determine the Clifford-Fourier Transform through sheaf theory in Lebesgue space sheaf. It is well known that the Fourier transform is successfully used to solving physical equations such as electrodynamics, fundamental problems of heat and mass transfer and wave equations concepts. For future development, we can apply the Clifford-Fourier Transform for possible determination of Wigner-Weyl-Moyal formalisms in quantum mechanics in phase space configuration.

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