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RESEARCH ARTICLE

NUMERICALLY EVALUATED NUMBER OF LEVEL CROSSINGS OF A RANDOM ALGEBRAIC POLYNOMIAL

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ABSTRACT

We study the expected number of real zeros of an algebraic polynomial of degree n, whose coefficients are independent Cauchy distributed random variables. We present a formula for the expected number, which has the advantage of being easy to use numerically. This approach shows that the error term involved in the asymptotic estimate for the expected number of real zeros with this class of distribution for the coefficients is 0(1).

INTRODUCTION

Let us consider a fixed probability space Ω, A, P, and let $a_j(w), j = 0,1.......$ be a sequence of independent random variables defined on Ω. Denote by $N_n(a,b)$= N(a,b) the number of real zeros of P(x) in the interval $a \leq 1 \leq b$ where

$$P_n(x, \mu) = P(x) = \sum_{j=0}^{n} a_j(w) x^j \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.1)$$

Assuming normal distribution with mean zero for the coefficients a_j, J=1…n. Kac (Ibragimov and Maslova, 1971) showed that for all sufficiently large n the mathematical implication of $N(-\infty, \infty)$, denoted by EN, is asymptotically equal to (2/π) of (n+1). There are several authors who have since considered the assumption of normality distributions for the coefficients, work whenever with different conditions for the coefficients or various types of polynomials. For instance, Sambandhan (Ibragimov and Maslova, 1971) showed that the above asymptotic formula remains invariant when the coefficients are dependent with correlation coefficient of any a_j and a_j, I,j=0,1…n as p_{ij}=p $0 < p < \frac{1}{2}, i \neq j.$ However, where p_{ij}=p $0 < p < \frac{1}{2}, i \neq j.$ is fixed, the asymptotic formula is reduced by half (Little Wood and Offord, 2005). This reduction is considered with the work of Ibragimov and Maslova (Farhamand, 1989), when they assume that coefficients a_j have a non-zero mean. Stevens (Christensen and Sambandhan, 1984) improved Kac's estimates for the case of the independent standard normal random coefficients and proved that the error term is 0(1). The lower bound improved by Christensen and Sambandham (Das, 1971). Very recently in interesting papers involving several new methods, Wilkins (Das, 1943; Rice, 1945) proved the upper bound further and showed the existence of numerical values of the first six coefficients in, an asymptotic expansion. In particular he obtained

$$EN \sim (2/\circledR)\log(n+1) + \sum_{j=0}^{5} D_j(n+1) + 0(n+1)^{-6}.$$

The values of the constants, D_j, j=0,1……5, are numerically evaluate and, unexpectedly, it is observed that D_j=0 for j=1,3,5.

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However, the problem is of a different level of difficulty when normality condition is relaxed. In this connection, the case when the belong to the domain of attraction of the normal law with means zero. Prob $(a_j \neq 0)$ >0, has been considered by Ibragimov and Maslova. They have shown that, for n sufficiently large $\mathrm{E}\,N(-\infty,\infty) \sim (2/®)(n+1)$. Therefore, for this case also the expected number of real zero remains as in the case of Kac referred to above. Hence, it is of special interest to seek a class of distributions, if it exists, such that one can obtain a different number of real zeros for the polynomial. To this Logan and Shepp (Kac, 1943) have shown that, when the coefficients a_j are independent with a common Cauchy distribution, for all sufficiently large n.

$$EN(-\infty,\infty) \sim C\log(\mathrm{n}+1),$$

where

$$\mathrm{C} = 8®^{-2}\int_0^x (xe^{-x})/(x-1+2e^{-x})dx.$$

This shows, since C≈0.741284 and 2/π=0.636620, that for the Cauchy distribution there are asymptotically more zeros. This is probably cause by the fact that in the Cauchy case the variance is infinite and therefore coefficients tend to be more spread out, and hence cancellation is easier. In this paper we re-examine Logan and Shepp's case and prove the following theorem.

Theorem

Let

$$\sim_p = \sim(p,x) = \sum_{j=0}^{p} x^j, \quad {}_p = \quad (p,x) = \sum_{j=0}^{p} jx^j \qquad (1.3)$$

$$¥_{\mathrm{mn}} = (\sim_n - 2\sim_m), ¶_{\mathrm{mn}} = ({}_n - 2 \quad {}_m), \qquad (1.4)$$

$$I_m = \{(1/¥_{\mathrm{mn}})\{[¶_{\mathrm{mn}} - m¥_{mn}]\log[¶_{\mathrm{mn}} - m¥_{mn}] - [¶_{\mathrm{mn}} - (m+1)¥_{mn}]$$
$$\log[¶_{\mathrm{mn}} - (m+1)¥_{mn}] - ¥_{mn}$$
$$= \sum_{m=0}^{n=1} \mathrm{I}_m$$

and

$$J(x) = (1/\sim_n)\{\quad n[\log \quad {}_n - 1] + (\sim_n n - \quad {}_n)[\log(\sim_n n - \quad {}_n) - 1]\}. \qquad (1.6)$$

Then

$$EN(-\infty,\infty) = 4®^{-2}\int_0^1 x^{-1}(I(x) - J(x))dx. \qquad (1.7)$$

As x=0 the integrand in (1.7) has a singularity whose nature must be understood when integrating it numerically. When numerical integration a performed to calculate $EN(0,1) = EN(-\infty,\infty)/4$, it is found that

$$EN(0,1) \sim (C/4)\log(n+1) + A_0 + A_2(n+1)^{-2} \qquad (1.8)$$

where C is given in (1.2), A_0=0.139783 and A_2=-0.057649.

This result suggests that, for this wider distribution of the coefficients, the error term is also 0(1) and , as in Wilkins (Rice, 1945) result, the term of $0(n+1)^{-1}$ vanishes. As the case of Cauchy distributed coefficients studied here has infinite variance and is not well behaved, one can perhaps conjecture that this property of the error term is common to all classes of distribution of coefficients.

A formula for the number of real zeros

The transformations $P(x) \to P(-x)$ and $\mathrm{P(x)} \to \mathrm{x}^{\mathrm{n}}\mathrm{P(1/x)}$ leave the coefficients distribution invariant. Therefore $EN(-\infty,\infty) = 4\,\mathrm{EN}\,(0,1)$, and we confine ourselves to the interval (0,1). By using Rice [7, p, 52] (see also [1,p,94], Logan and Shepp (Little Wood and Offord, 2005) obtained a formula for a mathematical expectation of the number of real zeros of P(x) as

$$EN(0,1) = \int_0^1 (1/®^2 x)dx\left[\int_0^{\mathrm{n}} \log \mathrm{A(u)du} + n - n\log A(n)\right]$$
$$+ (\quad {}_n/\sim_n)\log(n\sim_n - \quad {}_n)/ \quad {}_n$$
$$= ®^{-2}\int_0^1 x^{-1}dx\int_0^n \log g_n(u,x)du,$$

Where

$$A(u) = An(u,x) = \sum_{j=0}^{n} |u-j|x^j$$

and

$$g_n(u,x) = \frac{\sum_{j=0}^{n} |(u-j)|x^j}{\sum_{j=0}^{n} (u-j)x^j}$$

Let

$$I(x) = \int_0^n \log\left[\sum_{j=0}^{n} |u-j|x^j\right]dx$$
$$= \int_0^n \log\left[\sum_{j=0}^{n} |u-j|x^j\right]dx$$
$$\sin ce\ \mathrm{x} \geq 0.$$

Now, let m be the greatest contained in u; then, with α_p and β as in (1.3) and γ_{mn} and ζ_{mn} as in (1.4).

$$\sum_{j=0}^{n} |u-j|x^j = \sum_{j=0}^{n} |(u-j)|x^j + \sum_{j=0}^{n} |(u-j)|x^j$$
$$= u(2\sim_m - \sim_n) - (2 \quad {}_m - \quad {}_n),$$
$$= ¶_{mn} - u¥_{mn}.$$

Thus

$$I(x) = \int_0^n \log(¶_{\mathrm{mn}} - u¥_{mn})du.$$

Hence the integrand has discontinuous at integral values of u, it is conversant to write

$$I(x) = \sum_{m=0}^{n-1} \mathrm{I}_m$$

where

$$\mathrm{I}_m = \int_m^{m+1} \log\left(¶_{mn} - u¥_{mn}\right) du,$$

Such evaluates to the expression given for I_m in (1.5)

Let

$$J(x) = \int_0^n \log\left|\sum_{j=0}^{n} (u-j)x^j\right| du$$

$$= \int_0^n \log\left|\sim_n u - \ _n\right| du$$

The integrand of J(x) has a singularity at β_n/α_n, which lies between 0 and n after integration, the expression for J(x) in (1.6) is obtained. This result, along with (2.1) and (2.2) establishes (1.7).

NUMERICAL RESULTS

The integration in (1.7) is carried out numerically. As will be seen, a problem arises due to a singularity of the integrand at x=0. The nature of the singularity can be found by considering the behaviour of I(x) and as $x \to 0$. For small values of x, α_m =1, and

$¥_{mn} \cong -1, \ _0 = 0 \text{ and } \mathrm{i} = \mathrm{x}, \text{ while, if } \mathrm{m} > 0, \ _\mathrm{m} \cong x \text{ and } ¶_{\mathrm{mn}} = -x,$

Then, substituting these values into (1.5),

$$I_0 = -x \log x - 1,$$

$$I_m = (m+1)\log(m+1) - m \log m - 1 \qquad (\mathrm{m} > 0),$$

$$\sum_{\mathrm{m=1}}^{\mathrm{n-1}} \mathrm{I_m} = n \log n - (n-1)$$

$$J(x) = x \log x + n \log n - n,$$

where x^{-1}(I(x)-J(x))=-2log x. Since the integrand in (1.7) behaves as log x as $x \to 0$, the NAG library subroutine D01APF was used to integrate from 0 to 0.98. This routine requires that log x be used a weight function, so it is not appropriate for use up to x=1; the subroutine D01AJF was used for the remainder of the integration where the integration shows considerable variation on the interval (0.98, 1.0) for large values of n, the degree of the polynomial. Results have been obtained for a large number of values of n, and a useful selection is presented in Table-1. The final column of the table suggests that Er(n)= EN (0,1)-(C/4)log (n+1) approaches a limit $n \to \infty$. To investigate an appropriate behaviour of Er(n) more fully, NAG library subroutine G02DAF was used to fit a polynomial of the forms $A_0 + A_1(n+1)^{-1} + A_2(n+1)^{-2}$ to it; the data used were the 93 values of Er(n) for n=8,9,….100.

The values obtained for the coefficients were A0=0.139782, A1=0.000065 and A2=-0.058279. When considering the value of A1, it must be noted that the fact it is non-zero may be due to round off effects. The value found for A1 sugegsted the use of the lines regression subroutine G02CAF to fit a line with equation of the form that in (1.8) to the same data, giving

$$Er(n) = 0.139783 - 0.057649(n+1)^{-2}$$

Table 1.

n	*EN (0,1)*	*(C/4) log (n+1)*	*Er (n)*
10	0.583688	0.444381	0.139307
15	0.653378	0.513819	0.139559
20	0.703868	0.564214	0.139653
25	0.743493	0.603794	0.139698
30	0.776114	0.636390	0.139724
35	0.803841	0.664102	0.139739
40	0.827952	0.688203	0.139749
100	0.995057	0.855279	0.139777
150	1.069588	0.929808	0.139781
200	1.122596	0.982814	0.139782
325	1.212217	1.072434	0.139783
2000	1.548483	1.408700	0.139783

The correlation coefficient was found to be -0.999991, which again must be seen in the context of the rounding off the data. The result obtained in this wider case is consistent with that of KAC [5], when the coefficients are distributed normally, in that he finds the coefficients of (n+1) to be zero.

REFERENCES

Bharucha, A.T. and Sambandhan, M. 1986. Random Polynomial., Academic Press, New York, 1986.

Christensen, M. and M. Sambandhan, 1984. An improved lower bound for the expected number of real zeros of a random polynomial. *Stochastic Anal.*, 2, 431-436.

Das, M.K. 1971. Real zeros of a random sum of orthogonal polynomials, *Proc. Amer. Math. Soc.*, 27, 1 147-153.

Dunnage, J.E.A. 1968. The number of real zeros of a class of random algebraic polynomials (I), *Proc. London Math. Soc.*, (3) 18 439-460.

Farhamand, K. 1989. *Indian J. Pure Appl. Math.*, 20, 1-9.

Ibragimov, I. A. and N.B. Maslova, 1971. *Theor. Prob. Appl.*, 16, 228-48.

Ibragimov, L.A. and B. Maslova, 1971. On the expected number of real zeros of random polynomials II. Coeffcients with non zero means. *Theory Probab. Appl.*, 16, 485-493.

Kac, M. 1943. *Bull Am. Math. Soc.* 46, 314-20.

Kac, M. 1943. On the average number of real roots of a random algebraic equation. *Bull, Amer. Math. Soc.*, 49, 314-320.

Kac, M. 1943. On the average number of real roots of random algebraic equation, *Bull. Amer. Math. Soc.*, 49, 314-320.

Little Wood, J.E. and Offord, A.C. 2005. On the number of real roots of a random algebraic equation (II), *Proc. Cambridge phil. Soc.*, 35, 133-148.

Rice, S.O. 1945. *Bell System tech. J.*, 25, 46-156

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