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RESEARCH ARTICLE

ON THE CHARACTERIZATION OF LIE ALGEBRAS

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ABSTRACT

We present Cartan Subalgebra and root system. This leads to the study of Cartan matrix and Dynkin Diagrams. In this paper we study the relationship between Cartan matrix and Dynkin Diagrams.

INTRODUCTION

The theory of Lie algebra of a Lie group is considered to be one of the most important topics in temporary mathematics. It has wide applications in many branches of pure and applied mathematics, including symmetric spaces, particle physics, field theory, differential systems and applied mathematics. In fact, the theory of Lie algebra has been used extensively in the classification problems in many fields in Science. Representation theory of Lie algebra such as the ad joint representation is of prime importance in the classification scheme. In this article, we illustrated the theory of classification via the concept of the root system and the associated Cartan matrix and Dynkin diagram. We have provided some theorems that link the above concepts and give the relation between them.

Preliminaries

Definition

Let $\mathcal{G}$ be a Lie algebra over $\mathbb{C}$

(1) For any subalgebra$\mathfrak{b}$ of$\mathcal{G}$let

N (Б) = $\{X \in \mathcal{G}:(X, \mathfrak{b}) \subset \mathfrak{b}\}$,

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then $N(\mathfrak{b})$ is called the normalizer of $\mathfrak{b}$and it's the largest subalgebra of $\mathcal{G}$ which contains $\mathfrak{b}$as an ideal.

(2) A subalgebra $\mathfrak{b}$ of$\mathcal{G}$ is called a Cartan subalgebra if
i- It's nilpotent.
ii- N $(\mathfrak{b})$ =$\mathfrak{b}$

Definition (Root space)

Let $\mathcal{G}$ be an arbitrary semisimple Lie algebra over $\mathbb{C}$ and $\mathfrak{b}$denote an arbitrary fixed Cartan subalgebra. Let α be a linear function on complex vector space $\mathfrak{b}$ let

$$\mathcal{G}^{\alpha} = \{X \in \mathcal{G}: [H, X] = \alpha(H)X \; for \; all \; H \in \mathfrak{b}\}$$

if $\mathcal{G}^{\alpha} \neq 0$ then α is called a root and $\mathcal{G}^{\alpha}$ is called a root space.

Root system

Definition

Let V be a finite dimensional vector space over$\mathbb{R}$, and let (x, y) be a positive definite symmetric bilinear form on V, a finite subset R of nonzero vectors of V is called a root system if

(i) R spans V .

(ii) $\alpha \in R$ and $t\alpha \in R$ with $t \in \mathbb{R}$, then $t = \pm 1$.

(iii) $\alpha, \beta \in R$, then $2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer.

(iv) $\alpha, \beta \in R$, then $\beta - 2\frac{[(\alpha,\beta)]}{(\alpha,\alpha)}\alpha \in R$

The elements of R are called roots.

Definition (Root System Basis)

A subset $B \subset R$ is called a root system basis for the root system R in real finite dimensional vector space V if

(1) B is a vector space basis for V

(2) For any $\beta \in R$ we have $\beta = \sum_{i=1}^{n} m_i \alpha_i$

where

$B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and either all $m_i'^S$ are non-negative or they are all non-positive.

Definition

Given a basis $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ for a root system R in V , let $N(\alpha,\beta) = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}, \alpha, \beta \in R$

(i) The matrix $(N(\alpha_i, \alpha_j))$ is called the Cartan matrix of the root system R .

(ii) If $\alpha = \sum_{i=1}^{n} m_i \alpha_i \in R^+$, then $\sum_{i=1}^{n} m_i$ is called the height of α with respect to B .

Dynkin diagrams

Definition

Dynkin diagram of a root system R in V with basis $B = \{\alpha_1, \ldots, \alpha_n\}$ consist of a graph in $\mathbb{R}^2$ with n vertices labeled with $\alpha_1, \ldots, \alpha_n$ and $N(\alpha_i, \alpha_j)N(\alpha_j, \alpha_i)$ line segments joining the vertex α_i to the one α_j . Finally if $N(\alpha,\beta) \neq 0$ and $(\beta,\beta) > (\alpha,\alpha)$, draw an arrow on the line segments from the vertex of β to the vertex of α .

Definition

For each $\alpha \in R$ define the α symmetry of V

$$S_\alpha : V \to V$$

$$S_\alpha(x) = x - 2\frac{[(\alpha,x)]}{(\alpha,\alpha)}\alpha$$

Some theorems

Theorem (Sigurdur Helgason, 1962):

Every semisimple Lie algebra over $\mathcal{C}$ has a Cartan subalgebra.

Theorem (Sigurdur Helgason, 1962; Sigurdur Helgason, 1978)

Let $\mathcal{G}$ be a semisimple Lie algebra over $\mathcal{C}$, $\mathfrak{h}$ denote an arbitrary fixed Cartan subalgebra the set of all nonzero roots is a root system for $\mathcal{G}$

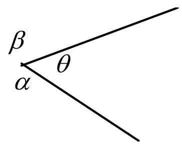
Theorem: Let $(,)$ be the usual inner product on $\mathbb{R}^2$. For any root system R and $\alpha, \beta \in R$ we have

$$\frac{2(\alpha,\beta)}{(\beta,\beta)} = 0, \pm 1, \pm 2, \pm 3 \cdot$$

Define $N(\alpha,\beta) = 2\frac{(\alpha,\beta)}{(\beta,\beta)}$

Proof:

Let θ denote the angle between α and β though as vectors in $\mathbb{R}^2$



$$(\alpha,\beta) = \sqrt{(\alpha,\alpha)}\sqrt{(\beta,\beta)}\cos\theta$$

$$(\alpha,\beta)^2 = (\alpha,\alpha)(\beta,\beta)\cos^2\theta$$

$$\cos^2\theta = \frac{(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)}$$

$$N(\alpha,\beta)N(\beta,\alpha) = 4\cos^2\theta \leq 4$$

Since $N(\alpha,\beta)$ and $N(\beta,\alpha)$ are all integers "definition (3.1)"

then $N(\alpha,\beta) = 0, \pm 1, \pm 2, \pm 3, \pm 4$.

Assume $N(\alpha,\beta) = 4$, then $N(\beta,\alpha) = 1$ and $\cos\theta = 1$

Thus $N(\beta,\alpha) = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} = 1$, $N(\alpha,\beta) = \frac{2(\alpha,\beta)}{(\beta,\beta)} = 4$

$\therefore 2(\alpha,\beta) = (\alpha,\alpha)$ and $2(\alpha,\beta) = 4(\beta,\beta)$

$$(\alpha - 2\beta, \alpha - 2\beta) = (\alpha, \alpha - 2\beta) - 2(\beta, \alpha - 2\beta) = (\alpha,\alpha) - 2(\alpha,\beta) - 2[(\beta,\alpha) - 2(\beta,\beta)]$$

$$= (\alpha,\alpha) - 2(\alpha,\beta) - 2(\beta,\alpha) + 4(\beta,\beta)$$

$$= (\alpha,\alpha) - 4(\alpha,\beta) + 4(\beta,\beta)$$

But $(\alpha,\alpha) = 2(\alpha,\beta), 4(\beta,\beta) = 2(\alpha,\beta)$

so $(\alpha - 2\beta, \alpha - 2\beta) = 2(\alpha,\beta) - 4(\alpha,\beta) + 2(\alpha,\beta) = 0$

then $(\alpha - 2\beta, \alpha - 2\beta) = 0$ (Property of inner product) implies

$\alpha = 2\beta$

This contradict if $\alpha \in R \wedge t\alpha \in R$, then $t = \pm 1$ "definition (3.1)"

Similarly for $N(\alpha,\beta) = -4$
then $N(\alpha,\beta) = 0 \pm 1, \pm 2, \pm 3$.

Lemma (Sigurdur Helgason, 1962):

If α and $\beta \in R$ with $\beta \neq \pm\alpha$ and $(\alpha,\beta) > 0$, then $\alpha - \beta \in R$.

Theorem:

If $\alpha, \beta \in R, \beta \neq \pm\alpha$, and p, q are the largest integers such that $\beta + p\alpha \in R$ and $\beta - q\alpha \in R$, then

(1) $\beta + k\alpha \in R$ for all integers k with $-q \leq k \leq p$

(2) $N(\beta,\alpha) = \dfrac{2(\alpha,\beta)}{(\alpha,\alpha)} = q - p$

Proof:

(1) The result is obvious unless $p > 1$ or $q > 1$. Assume that $p > 1$, then $-3 \leq N(\beta,\alpha) = \dfrac{2(\alpha,\beta)}{(\alpha,\alpha)} \leq 3$ "theorem (5.3)" , so that $\frac{-3(\alpha,\alpha)}{2} \leq (\alpha,\beta) \leq \frac{3(\alpha,\alpha)}{2}$ and for $k \geq 2$ we have $k(\alpha,\alpha) - \frac{3(\alpha,\alpha)}{2} \leq (\alpha,\beta) + k(\alpha,\alpha) \leq k(\alpha,\alpha) + \frac{3(\alpha,\alpha)}{2}$
Then $0 < (k - 3/2)(\alpha,\alpha) \leq (\alpha,\beta) + k(\alpha,\alpha) \leq (k + 3/2)(\alpha,\alpha)$ and we have $(\alpha, \beta + k\alpha) = (\alpha,\beta) + k(\alpha,\alpha) > 0$. By lemma (5.4) if $\beta + k\alpha \in R$, then $\beta + k\alpha - \alpha = \beta + (k-1)\alpha \in R$.

Thus $\beta, \beta+\alpha, \beta+2\alpha, \ldots\ldots, \beta+p\alpha \in R$,. by considering $-\alpha$ instead of α we find $\beta - q\alpha$, $\beta - (q-1)\alpha, \ldots, \beta \in R$.
(2) Since $S_\alpha(\beta + k\alpha) = \beta + k\alpha - \dfrac{2[(\alpha, \beta + k\alpha)]}{(\alpha,\alpha)}\alpha$

$= \beta + k\alpha - N(\beta,\alpha)\alpha - 2k\alpha$

$= \beta - \alpha(N(\beta,\alpha) + k)$

We must have $S_\alpha(\beta + p\alpha) = \beta - q\alpha$ "symmetry" …(i)
and $S_\alpha(\beta + p\alpha) = \beta - (p + N(\beta,\alpha))\alpha$ …….(ii)
from (i) and (ii)

$N(\beta,\alpha) = q - p$

Theorem

All roots of a root system R can be determined from a basis B for R and the Cartan matrix for R with respect to B.

Proof:

We will proceed by induction on the height of roots to find all roots of R^+. The roots of height one i.e., $\sum m_i = 1$ are just those on B. Assume we know the roots of R^+ of height k and we wish to find the root of height $(k+1)$, every roott of this height is of the form $\alpha + \alpha_i$ (1) with $\alpha \in R^+$ of height k and $\alpha_i \in B$. The Cartan matrix allows us to compute $N(\alpha,\alpha_i)$. But $N(\alpha,\alpha_i) = q - p$ where $\alpha + k\alpha_i \in R$ for $-q \leq k \leq p$ "theorem (5.5)". By induction hypothesis q is known, so we are able to determine p and decide whether $\alpha + \alpha_i \in R$ or not.

Corollary

Two root systems with basis such that their Cartan matrices are identical are isomorphic.

Theorem

The Dykin diagram of a root system with basis $B = \{\alpha_1, \ldots, \alpha_n\}$ completely determines the corresponding Cartan matrix.

Proof:

We want to show that $N(\alpha_i, \alpha_j)$ can be determined for $i \neq j$

(1) If vertices α_i and α_j are not joined, then
$N(\alpha_i, \alpha_j) = 0$

(2) If they are joined by single line, then
$N(\alpha_i, \alpha_j) = N(\alpha_j, \alpha_i) = -1$

(3) If they are joined by two lines i.e.,
$N(\alpha_i, \alpha_j)N(\alpha_j, \alpha_i) = 2$.

Assume $(\alpha_j, \alpha_j) > (\alpha_i, \alpha_i)$, then

$$-N(\alpha_i, \alpha_j) = -2\frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} < -2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = -N(\alpha_j, \alpha_i)$$

$N(\alpha_j, \alpha_i) = -2$, $N(\alpha_i, \alpha_j) = -1$

(4) If $N(\alpha_i, \alpha_j)N(\alpha_j, \alpha_i) = 3$ and $(\alpha_j, \alpha_j) > (\alpha_i, \alpha_i)$

By the same manner

$N(\alpha_i, \alpha_j) = -1$ and $N(\alpha_j, \alpha_i) = -3$

Conclusion

For Semi simple Lie algebra$\mathcal{G}$ over $\mathbb{C}$ with Cartan Subalgebra and root system we can get Cartan matrix "theorem (5.3)", Cartan matix determine Dynkin diagram by definition. On the other hand Dynkin diagram of a root system R with basis B completely determine Cartan matrix "theorem (5.8)". From Cartan matrix and B we can get root system "theorem (5.6)", which determine Lie algebra up to an isomorphism (Sigurdur Helgason, 1962).

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