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RESEARCH ARTICLE

ACCURATE SOLUTIONS OF FIRST ORDER NONLINEAR INITIAL VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS WITHTHERUNGE - KUTTA - FEHLBERG METHOD

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ABSTRACT

This paper applies the fifth order Runge-Kutta-Fehlberg method for solving nonlinear first orderinitial value problem. The proposed method is quite efficient and practically well suited for solving these problems. In order to verify the accuracy, we compare numerical solutions with the exact solutions. The numerical solutions are in good agreement with the exact solutions. The stability analysis of the method has been investigated. Three model examples are given to demonstrate the reliability and efficiency of the methods. The proposed method also compared with some previously existing literatures and shows betterment results.

INTRODUCTION

Due to the advancement in the field of computational mathematics numerical methods are most widely utilized to solve the equation rising in the field of applied mathematical science, engineering and technology. Many problems of mathematical physics can be started in the form of differential equations. Numerical methods are commonly used for solving mathematical problems that are formulated in science and engineering where it is difficult or even impossible to obtain exact solutions. Only a limited number of differential equations can be solved analytically (Amirul Islam, 2015). Even then there exist a large number of ordinary differential equations whose solutions cannot be obtained in closed form by using well-known analytical methods, where we have to use the numerical methods to get the approximate solution of a differential equation under the prescribed initial condition. There are many types of practical numerical methods for solving first order initial value problems for ordinary differential equations. Such as a new third order inverse Runge-Kutta method by (Agbeboh and Omonkaro, 2010) Euler and Runge Kutta Methods by (Amirul Islam, 2015), Tau-Collocation Approximation Approach by (James *et al*., 2016) and Fourth order Runge-Kutta method by (Gemachis and Tesfaye, 2016).

Even if, much attention has been attracted and different methods have been applied to find the numerical solution of first order non-linear initial value problems, yet there is lack of accuracy. Therefore, we use Runge-KuttaFlehberg method whichis the fifth order convergent that gives the more accurate numerical solution for the proposed problem.

Application of the method

Consider the nonlinear initial value problem of the form:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where x_0, y_0 are given constants and with function $y(x)$ is unknown.

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For the independent variable $x \in [x_0, x_f]$, divide the interval $[x_0, x_f]$ into N equal subintervals of mesh length h and the mesh points given by $x_i = x_0 + ih,\ i = 1, 2, \ldots, N-1$. To solve the problem given in Eq. (1), we apply the single step method that requires information about the solution at x_i to calculate x_{i+1} (Gemechis and Tesfaye, 2016). From one of the single step methods and the family of Runge Kutta methods, the general numerical solution of Eq. (1) using the fifth order Runge Kutta method given as:

$$y_{i+1} = y_i + \sum_{i=1}^{5} w_i k_i$$

where

$$k_i = hF(x_i + c_i h\ ,\ y_i + \sum_{j=1}^{4} a_{ij} k_{j},\ z_i + \sum_{j=1}^{4} a_{ij} m_j)$$

(John and Kurtis, 2004), was presented the Runge - Kutta - Fehlberg method to solve a first order initial value problems of the form of Eq. (1), which is given by:

$$y_{n+1} = y_n + \frac{16}{135}k_1 + \frac{6656}{12{,}825}k_3 + \frac{28{,}561}{56{,}430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6 \qquad \ldots\ldots(3)$$

Where $k_1 = hf(x_n, y_n)$,

$$k_2 = hf(t_n + \frac{h}{4}, y_n + \frac{k_1}{4}),$$

$$k_3 = hf(t_n + \frac{3}{4}h, y_n + \frac{3}{32}k_1 + \frac{9}{32}k_2),$$

$$k_4 = hf(t_n + \frac{12}{13}h, y_n + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3),$$

$$k_5 = hf(t_n + h,\ y_n + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4)$$

$$k_6 = hf(t_n + \frac{1}{2}h, y_n - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5)$$

In the determination of the parameters, since the terms are up to $o(h^5)$ be compared, the truncation error is $o(h^6)$ and the order of method is $o(h^5)$, (Grewal, 2002, Jain *et al*, 2007 and John and Kurtis, 2004).

Stability Analysis

To see the stability of the present method, the nonlinear function of Eq. (1) can be linearized by expanding the function $f(x, y)$ in Taylor series about the point (x_0, y_0), and truncating it after the first derivative:

$$y' = f(x_0, y_0) + (x - x_0)\frac{\partial f}{\partial x}(x_0, y_0) + (y - y_0)\frac{\partial f}{\partial y}(x_0, y_0) \qquad \ldots\ldots(4)$$

which can be re-written as:

$$y' = y\frac{\partial f}{\partial y}(x_0, y_0) + y_0' - y_0\frac{\partial f}{\partial y}(x_0, y_0) \qquad \ldots\ldots(5)$$

$$y' = \lambda y + c \qquad \ldots\ldots(6)$$

which is linear with respect to variable y.

where, $\lambda = \frac{\partial f}{\partial y}(x_0, y_0)$ and $c = y_0' - y_0 \frac{\partial f}{\partial y}(x_0, y_0)$

Let the test differential equation of linear differential Eq. (6) is of the form:

$$w' = \lambda w, \quad w(x_0) = w_0 \quad \text{.......................(7)}$$

where, λ is a constant, and has its solution in the form of,

$$w(x) = w(x_0)e^{\lambda(x-x_0)}$$

$\Rightarrow y(x_n) = y(x_0)e^{\lambda nh} = y_0(e^{\lambda h})^n$, at $x_n = x_0 + nh$ and taking $w = y$ ....(8)

Now, by considering the formula of Runge-KuttaFlehberg method in Eq. (3), we have:

$$k_1 = hf(x_n, y_n) = \lambda h y_n$$

$$k_2 = \lambda h\left(1+\frac{1}{4}\lambda h\right)y_n = \left(\lambda h+\frac{1}{4}(\lambda h)^2\right)y_n$$

$$k_3 = \lambda h\left(1+\frac{3}{32}\lambda h+\frac{9}{32}\left(\lambda h+\frac{1}{4}(\lambda h)^2\right)\right)y_n = \left(\lambda h+\frac{3}{8}(\lambda h)^2+\frac{9}{128}(\lambda h)^3\right)y_n$$

$$k_4 = \left(\lambda h+\frac{12}{13}(\lambda h)^2+\frac{72}{169}(\lambda h)^3+\frac{513}{2197}(\lambda h)^4\right)y_n$$

$$k_5 = \left(\lambda h+(\lambda h)^2+\frac{1}{2}(\lambda h)^3+\frac{5}{12}(\lambda h)^4-\frac{5}{104}(\lambda h)^5\right)y_n$$

$$k_6 = \left(\lambda h+\frac{1}{2}(\lambda h)^2+\frac{1}{8}(\lambda h)^3-\frac{1}{24}(\lambda h)^4-\frac{11}{1248}(\lambda h)^5+\frac{11}{832}(\lambda h)^6\right)y_n$$

On substituting the values of $k_1, k_2, ..., k_6$, in Runge-Kutta-Flehberg method of Eq. (3), we obtain:

$$y_{n+1} = \left\{1+\lambda h+\frac{1}{2}(\lambda h)^2+\frac{1}{6}(\lambda h)^3+\frac{1}{24}(\lambda h)^4+\frac{1}{120}(\lambda h)^5+\frac{1}{2080}(\lambda h)^6\right\}y_n$$

$$y_{n+1} = E(\lambda h)y_n \quad \text{.......................(9)}$$

where:

$$E(\lambda h) = 1+\lambda h+\frac{1}{2}(\lambda h)^2+\frac{1}{6}(\lambda h)^3+\frac{1}{24}(\lambda h)^4+\frac{1}{120}(\lambda h)^5+\frac{1}{2080}(\lambda h)^6$$

From Eq. (8), it is easily observed that the exact value of $y(x_n)$ increases for the constant $\lambda > 0$ and decreases for $\lambda < 0$ with the factor $e^{\lambda h}$. While, from Eq. (9) the approximate value of y_n increases or decreases with the factor of $E(\lambda h)$.

If $\lambda h > 0$, then $e^{\lambda h} \geq 1$, so the fifth order Runge-Kutta-Fehlberg method (RKF5)is relatively stable. If $\lambda h < 0$ $(i.e, \lambda < 0,$ since $h > 0)$, then the interval of absolutely stable is $-3.677 < \lambda h < 0$.

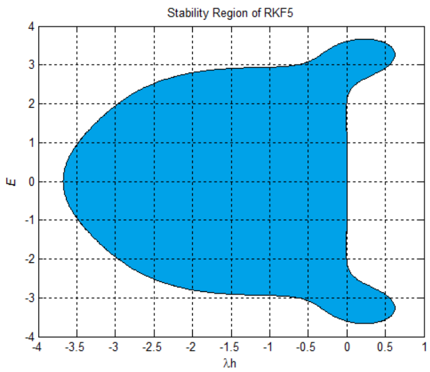

Figure 1. The stability region of Runge – Kutta - Fehlberg method

NUMERICAL EXAMPLES AND RESULTS

To demonstrate the applicability of the method, three model examples of nonlinear first order initial value problems have been considered. These examples have been chosen because they have been discussed in different articles by different scholars.

Example 1: Consider the following non-linear initial value problem:

$$\frac{dy}{dx}=\frac{-1}{1+x}+y(x)-y^2(x), \quad 0\le x\le 1$$

with the initial condition $y(0)=1$ and exact solution is given as: $y(x)=\frac{1}{1+x}$.

Table 1. Pointwise absolute errors at different values of the mesh sizes for Example 1

t	$N=10$	$N=40$	$N=100$	$N=400$
The Present method				
0.1	8.7551e-09	6.4695e-12	6.3838e-14	1.1102e-16
0.2	1.1965e-08	9.0432e-12	8.9373e-14	5.5511e-16
0.3	1.3021e-08	9.9920e-12	9.9143e-14	6.6613e-16
0.4	1.3232e-08	1.0258e-11	1.0181e-13	5.5511e-16
0.5	1.3132e-08	1.0248e-11	1.0170e-13	7.7716e-16
0.6	1.2942e-08	1.0145e-11	1.0092e-13	3.3307e-16
0.7	1.2757e-08	1.0027e-11	1.0003e-13	7.7716e-16
0.8	1.2613e-08	9.9324e-12	9.9254e-14	2.2204e-15
0.9	1.2524e-08	9.8742e-12	9.8810e-14	2.4425e-15
1.0	1.2495e-08	9.8581e-12	9.8699e-14	7.2164e-16
(Gemechis and Tesfaye ,2016)				
0.1	3.8296e-07	1.2712e-09	3.1445e-11	1.2057e-13
0.2	5.7951e-07	1.9396e-09	4.8062e-11	1.8452e-13
0.3	6.8133e-07	2.2939e-09	5.6914e-11	2.1860e-13
0.4	7.3394e-07	2.4816e-09	6.1630e-11	1.8452e-13
0.5	7.6091e-07	2.5808e-09	6.4137e-11	2.4647e-13
0.6	7.7483e-07	2.6340e-09	6.5490e-11	2.5280e-13
0.7	7.8257e-07	2.6648e-09	6.6278e-11	2.5668e-13
0.8	7.8799e-07	2.6865e-09	6.6837e-11	2.5946e-13
0.9	7.9326e-07	2.7069e-09	6.7358e-11	2.6190e-13
1.0	7.9961e-07	2.7304e-09	6.7954e-11	2.6240e-13

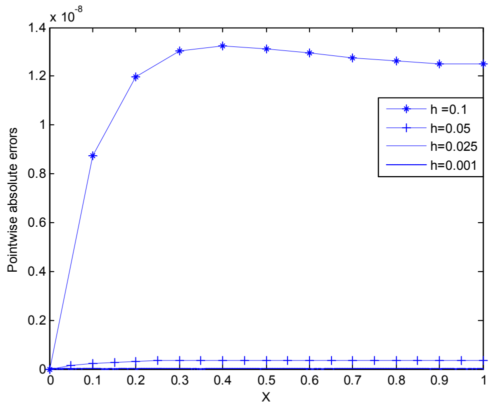

Figure 2. Pointwise absolute errors at different values of mesh sizes for Example 1

Example 2: Consider the initial value problem

$(1+x)y'(x)+y(x)=0,\ y(0)=1$. The exact solution is: $y(x)=\dfrac{1}{1+x}$.

Table 2. Numerical Results at *h = 0.1*for Example 2

	The present Method			(James et. al., 2016)	
x	Exact Solutions	Approximate Solution	Errors	Approximate Solution	Errors
0.1	9.0909e-01	9.0909e-01	0.0000e+00	0.9090418	4.19133e−05
0.2	8.3333e-01	8.3333e-01	0.0000e+00	0.8332214	1.1195e−04
0.3	7.6923e-01	7.6923e-01	2.2204e-16	0.7691735	5.7276e−05
0.4	7.1429e-01	7.1429e-01	2.2204e-16	0.7143198	3.4081e−05
0.5	6.6667e-01	6.6667e-01	2.2204e-16	0.6667378	7.1164e−05
0.6	6.2500e-01	6.2500e-01	2.2204e-16	0.6250298	2.9821e−05
0.7	5.8824e-01	5.8824e-01	1.1102e-16	0.5881915	4.3800e−05
0.8	5.5556e-01	5.5556e-01	1.1102e-16	0.5554809	7.4633e−05
0.9	5.2632e-01	5.2632e-01	2.2204e-16	0.5265873	2.8445e−05
1.0	5.0000e-01	5.0000e-01	2.2204e-16	0.5000000	0.0000e+00

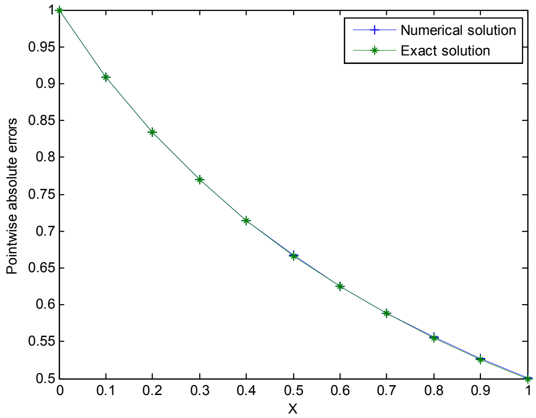

Figure 3. The behavior of exact and numerical solution for Example 2

Example 3: Consider the initial value problem:

$$y'(x) = x^2 - y(x); \quad y(0) = 1, \text{ on } 0 \le x \le 1.$$

The exact solution of the given problem is given by: $y(x) = x^2 - 2x - e^{-x} + 2$

Table 3. The comparison of pointwise absolute errors of Example 3.

x_i	$h = 0.1$	$h = 0.05$	$h = 0.0125$
The Present method			
0.1	5.417648551997445e-10	1.708844177272795e-11	1.654232306691483e-14
0.2	1.116527981182003e-09	3.509126322853717e-11	3.441691376337985e-14
0.3	1.713103103817559e-09	5.368028244134848e-11	5.240252676230739e-14
0.4	2.322133707544083e-09	7.258083023486961e-11	7.127631818093505e-14
0.5	2.935846676876963e-09	9.156508884444747e-11	8.926193117986259e-14
0.6	3.547835691897205e-09	1.104460967127352e-10	1.080247002960277e-13
0.7	4.152870825002708e-09	1.290711981738468e-10	1.261213355974178e-13
0.8	4.746733672789105e-09	1.473187127842834e-10	1.436628593864953e-13
0.9	5.326071028655122e-09	1.650914960293903e-10	1.604272270583351e-13
1.0	5.888270204756907e-09	1.823146078550053e-10	1.770805724277125e-13
(Amirul Islam Md., 2015)			
0.1	1.263692928077375e-7	7.78998954231724e-9	3.010358629040866e-11
0.2	2.485128776097416e-7	1.53062285068728e-8	5.911260370083937e-11
0.3	3.660906238156514e-7	2.25302732026477e-8	8.696288134046881e-11
0.4	4.788654034415529e-7	2.94496684816181e-8	1.136121197120587e-10
0.5	5.866865685488776e-7	3.60569771817864e-8	1.390368931097896e-10
0.6	6.894756376940592e-7	4.23489353584827e-8	1.632280977048594e-10
0.7	7.872139266007494e-7	4.83257271977067e-8	1.861915066569963e-10
0.8	8.799318765850828e-7	5.39903569629629e-8	2.079408867317056e-10
0.9	9.676998754537536e-7	5.93481118693617e-8	2.284998856794118e-10
1.0	10.50620377252009e-7	6.44060995647066e-8	2.478974803210576e-10

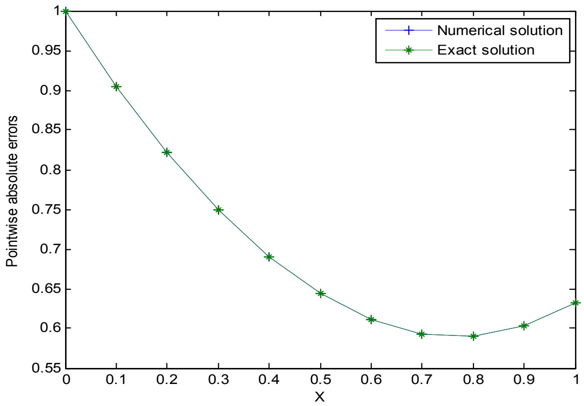

Figure 4. The behavior of exact and numerical solution for Example 3

Example 4: Consider the initial value problem:

$y'(x) = \cos^2(x) - \tan(x)y(x), \quad 0 \le x \le 1$, with the initial condition: $y(0) = 2$.

The exact solution of this problem is: $y(x) = 2\cos(x) + \cos(x)\sin(x)$.

Table 4: Absolute Maximum errors of Example 4, for $h = 0.05$.

x_i	Exact Solution	Absolute Errors	
		(Osama, et. al., 2015)	The Present Method.
0.05	2.0474	2.9×10-3	1.1988e-11
0.15	2.1253	1.3×10-3	4.6237e-11
0.25	2.1775	2.2×10-3	8.5721e-11
0.35	2.2009	1.7×10-3	1.1499e-10
0.45	2.1926	1.9×10-3	1.0961e-10
0.55	2.1507	1.6×10-3	3.0572e-11
0.65	2.0739	1.5×10-3	1.8597e-10
0.75	1.9621	1.2×10-3	6.4858e-10
0.85	1.8158	9.7×10-4	1.5518e-09
0.95	1.6365	7.3×10-4	3.2693e-09

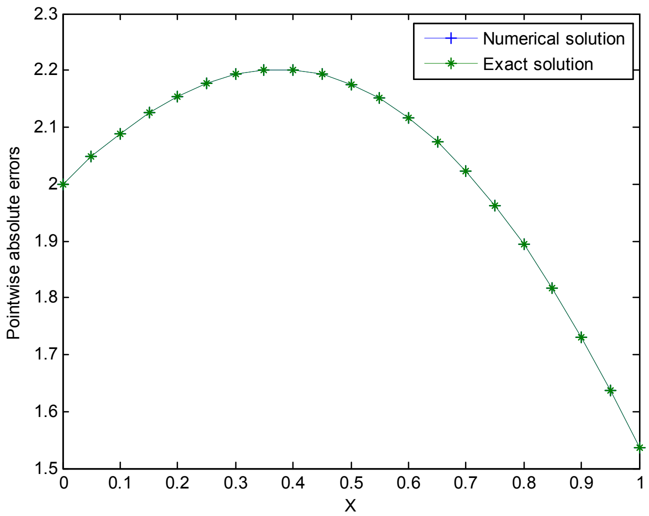

Figure 5. The relationship between the exact and numerical solution for Example 4

DISCUSSION AND CONCLUSION

In this paper, the Runge Kutta Fehlberg method has been presented for solving nonlinear first order initial value problems. Its stability analysis has been investigated and the method is absolutely stable in the interval $-3.677 < \lambda h < 0$. To validate the applicability of the proposed method, four model examples have been considered and solved. Point wise absolute errors are obtained by using MATLAB software.As it can be observed from the numerical results presented in Tables 1-4 and graphs (Figs. 3 - 5), the present method approximates the exact solution very well than the results presented by different scholars. Moreover, it can be observed the pointwise errors decreases rapidly as the number of mesh sizes decreases. Therefore, theRunge Kutta Fehlberg method is stable, more accurate and convergent method to find the numerical solution of the nonlinear first order initial value problems.

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