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RESEARCH ARTICLE

A-CONTRACTIONS RELATIVE TO A WEAK S-METRIC AND COMMON FIXED POINT THEOREMS

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ABSTRACT

Recently the authors have introduced the notion of a weak S-metric p on a S-metric space . Akram et.al have defined a class A of functions α: ${}_{+}^{3} \to {}_{+}^{3}$, where ${}_{+}$ =[0,∞). In this paper we consider a new class $\zeta_\alpha{}^p$ of pairs of self maps of a S-metric space with a weak S-metric p, where $\alpha \in A$ and prove two common fixed point theorems for members in that class , analogous to the results proved for metric spaces by Akram et.al.

INTRODUCTION

Sedghi, Shobe and Aliouche [2] have introduced S-metric spaces as a generalization of metric space as follows:

Definition ([2]. Definition 2.1) Let X be a non empty set. A *S-metric* on X is a function $S: X^3 \to [0, \infty)$ satisfying the conditions given below for each $x, y, z, a \in X$:

(a) $S(x, y, z) \geq 0$
(b) $S(x, y, z) = 0$ if and only if $x = y = z$.
(c) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

A set X equipped with a S-metric on it is called a *S-metric space* and it is denoted by (X, S).

1.2 Examples

(i) Let (X, d) be a metric space. Define $S_d : X^3 \to [0, \infty)$ by
$S_d(x, y, z) = d(x, z) + d(y, z) + d(x, y)$ for $x, y, z \in X$ and verify that (X, S_d) is a S-metric space.
(ii) Let $X_0 = [0,1)$ and S: $X_0^3 \to [0, \infty)$ be defined by $S(x, y, z) = |x \quad z| + |y \quad z|$ for $x, y, z \in X_0$.

Then (X_0, S) is a S-metric space. For other examples see [2].

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The definition and basic results given below, proved in [2], are needed in our discussion:

Let (X,S) be a S-metric space and $\{x_n\}$ be a sequence in it.

(1.3) If there is a x ∈X such that to each $\in > 0$ there is a natural number n_0 with $S(x_n, x_n, x) < \in$ for all n≥ n_0, then $\{x_n\}$ is said to *converge* to x in (X, S) ; and write $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$. ([2], Definition 2.8 (3)).

(1.4) If to each $\in > 0$ there is a natural number n_0 such that $S(x_n, x_n, x_m) < \in$ for all $m > n \geq n_0$ then $\{x_n\}$ is said to be a *Cauchy sequence* in (X, S) . ([2], Definition 2.8 (4)).

(1.5) If $\{x_n\}$ converges to x in (X, S) then x is unique. ([2].Lemma 2.10)
(1.6) If $\{x_n\}$ converges in (X, S) then $\{x_n\}$ is a Cauchy sequence. ([2].Lemma 2.11)
(1.7) (X, S) is said to be *Complete* if every Cauchy sequence in it converges to a point in X . ([2], Definition 2.8(5)).
(1.8) $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$ ([2], Lemma 2.5)

(1.9)If $\{x_n\}$ and $\{y_n\}$ are sequences in X converging respectively to x and y then $\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y)$ ([2], Lemma 2.11) .

Recently the authors [3] have introduced a weak S-metric p on a S-metric space as follows:

1.10. Definition: Suppose (X, S) is a S-metric space. A w*eak S-metric* on X is a function $p: X^3 \to [0, \infty)$ such that

(a′) $p(x, y, z) \leq p(a, a, x) + p(a, a, y) + p(a, a, z)$ for all $x, y, z, a \in X$
(b′) for each $x \in X$ the function, $p(x, x, .): X \to [0, \infty)$ is continuous on X. That is, for every sequence $\{y_n\}$ in X with $\lim_{n\to\infty} y_n = y$ we have $p(x, x, y) = \lim_{n\to\infty} p(x, x, y_n)$.
(c') to each $\in > 0$ there is a $\delta > 0$ such that $p(a, a, x) < \delta, p(a, a, y) < \delta$ *and* $p(a, a, z) < \delta$ for some $a \in X$ imply that $S(x, y, z) < \in$.

Examples

(i) Suppose (X, S) is a S-metric space. If $p: X^3 \to [0, \infty)$ is defined by $p(x, y, z) = S(x, y, z)$ for $x, y, z \in X$ then p is a weak S-metric on X. That is , *every S-metric on a set X is a weak S-metric on it.*
(ii)
(ii) Suppose (X_0, S) be the S-metric space given in Example 1.2(ii) . Define $p: {X_0}^3 \to [0, \infty)$ by $p(x, y, z)$ = y+2z for $x, y, z \in X_0$, and it can be verified that p is a weak S –metric on X_0.

1.12 .Remark. If p is a weak S-metric on a S-metric space (X, S) then $p(x, y, z) = 0$ need not imply x= y = z ; and that p (x, x, x) need not be zero. Also p (x, x, y) and p (y, y, x) need not be equal for all $x, y \in X$ (in contrast to (1.8)).

For instance , note that $p(a, 0,0) = 0$ for all $a \in X_0$ in Example 1.11(ii).

1.13.Definition.

Let $_+$ = [0,∞) and A be the class of all functions α: ${}_+^{\,3} \to {}_+$ satisfying the conditions

(i) α is continuous ${}_+^{\,3}$ (with respect to the Euclidean metric on ${}_+^{\,3}$)
(ii) $a \leq \alpha(a, b, b)$ or $a \leq \alpha(b, a, b)$ *or* $a \leq \alpha(b, b, a)$ for all $a, b \in {}_+$ implies $a \leq kb$
for some $k \in [0,1)$
1.14 Examples
(i) If $\alpha(u, v, w) = L\, u$ for $u, v, w \in {}_+$, where 0≤ $L < 1$ then $\alpha \in A$
(ii)If $\alpha(u, v, w) = k(v + w)$ for $u, v, w \in {}_+$, where 0≤ $k < \frac{1}{2}$ then $\alpha \in A$
(iii) If $\alpha(u, v, w) = k \max\{v, w\}$ for $u, v, w \in {}_+$, where 0≤ $k < 1$ then $\alpha \in A$
(iv) If $\alpha(u, v, w) = a\, u + b\, v + c\, w$ for $u, v, w \in {}_+$,where $a + b + c \leq 1$,$a \geq 0, b \geq 0, c \geq 0$ then $\alpha \in A$.

The class A was first considered by Akram,Zafar and Siddique[1] to define a class of contractions on a metric space called *A-contractions*. They proved that several known contractions are all A-contractions for suitably chosen $\alpha \in A$. Now using the class A we define a class of pairs of self maps on a S-metric space equipped with a weak S –metric .

***1.15 Definition*:** Suppose (X,S) is a S-metric space and p is a weak S-metric on X .If an ordered pair (F,G) of self maps of X is such that (1.15)′ $p(Fx, Fy, Gz) \leq \alpha(p(x, x, y), p(Fx, Fx, x), p(Gy, Gy, y))$ for all $x, y \in X$ holds for some $\alpha \in A$ then we write (F,G) $\in {\zeta_\alpha}^p$.

For example, if f is a self maps of (X, S) and α is the function defined in Example 1.14(i) ,then $(f, f) \in \zeta_\alpha{}^p$ is equivalent to (1.16) $S(fx, fx, fy) \leq L\, S(x, x, y)$ for all $x, y \in X$ holds, where $0 \leq L < 1$. Observe that any f satisfying (1.16) is called a *Contraction* in [2] (Definition 2.13).

1.17 Remark: If the weak S –metric p on a S-metric space (X, S) is such that $p(x, x, y) = p(y, y, x)$ for all $x, y \in X$ then it is easy to see that$(F, G) \in \zeta_\alpha{}^p \Leftrightarrow (G, F) \in \zeta_\alpha{}^p$. In particular ,if $p = S'$,another S-metric on (X, S) then ,in view of (1.8) for S', ,it follows $(F, G) \in \zeta_\alpha{}^{s'} \Leftrightarrow (G, F) \in \zeta_\alpha{}^{s'}$.

The following lemma proved in [3] is required for our discussion.

1.18 Lemma

Suppose (X, S) is a S-metric space and p is a weak S-metric on it .Suppose $\{x_n\}$ is a sequence in X ; $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $_+$ such that $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$; and $x, y, z \in X$. Then

(i) $p(x_n, x_n, x) \leq \alpha_n$ and $p(x_n, x_n, y) \leq \beta_n$ for all $n \geq 1$ imply x =y .In particular , $p(z, z, x) = p(z, z, y) = 0$ imply x=y.

(ii)$p(x_n, x_n, x_m) \leq \alpha_n$ for all m> n≥ 1 implies $\{x_n\}$ is a Cauchy sequence in(X, S).

In this paper we prove two common fixed point theorems for members of $\zeta_\alpha{}^p$ of a complete S-metric space with a weak S-metric on it.

Common fixed point theorems

In this section we prove two common fixed point theorems for self maps of a complete S-metric space with a weak S-metric on it.
2.1 Theorem : Suppose (X. S) is a complete S –metric space with a weak S-metric p defined on it satisfying

(2.1)′ $S(x, y, z) \leq p(x, x, y)$ for all $x, y \in X$.

Suppose f and g are self maps of X such that

(2.1)″ g is continuous on (X, S)

And

(2.1)‴ $(f, g) \in \zeta_\alpha{}^p$ and $(g, f) \in \zeta_\alpha{}^p$ for some $\alpha \in A$.

Then

(i) for any $x_0 \in X$,the sequence $\{x_n\}_{n=1}^{\infty}$ defined for n≥ 1 by $x_n = f\, x_{n-1}$ or $g\, x_{n-1}$ according as n is even or odd , converges to a point z ∈ X ,provided $\lim_{n\to\infty} p(x_n, x_n, x_n) = 0$

And

(iii) z is the unique common fixed point for f and g provided $p(z, z, z)=0$.

Proof:

In view of *(* 2.1)‴ we have an $\alpha \in A$ such that

(2.2) $p(fx, fx, gy) \leq \alpha(p(x, x, y), p(fx, fx, x), p(gy, gy, y))$ for all $x, y \in X$

and

(2.3) $p(gx, gx, fy) \leq \alpha(p(x, x, y), p(gx, gx, x), p(fy, fy, y)$ for all $x, y \in X$ Let $x_0 \in X$ and $\{x_n\}_{n=1}^{\infty}$ be defined by

$$x_n = \begin{cases} fx_{n-1}, \text{if n is even} \\ gx_{n-1}, \text{if n is odd} \end{cases}$$

be such that

(2.4) $\lim_{n\to\infty} p(x_n, x_n, x_n) = 0$

Write $A_n = p(x_{n+1}, x_{n+1}, x_n)$ for $n \geq 1$ and also $A_n = P_n$ or Q_n according as n is odd or even.

If n is odd, then, by (2.2), $P_n = p(x_{n+1}, x_{n+1}, x_n) = p(fx_n, fx_n, gx_{n-1})$

$$\leq \alpha(p(x_n, x_n, x_{n-1}), p(fx_n, fx_n, x_n), p(gx_{n-1}, gx_{n-1}, x_{n-1}))$$
$$= \alpha(p(x_n, x_n, x_{n-1}), p(x_{n+1}, x_{n+1}, x_n), p(x_n, x_n, x_{n-1}))$$
$$= \alpha (Q_{n-1}, P_n, Q_{n-1}),$$

so that ,by (ii) of Definition 1.13, we get $P_n \leq k_1 Q_{n-1}$ for some $k_1 \in [0,1)$
and if n is even then , by (2.3),

$$Q_n = p(x_{n+1}, x_{n+1}, x_n) = p(gx_n, gx_n, fx_{n-1})$$
$$\leq \alpha(p(x_n, x_n, x_{n-1}), p(gx_n, gx_n, x_n), p(fx_{n-1}, fx_{n-1}, x_{n-1}))$$
$$= \alpha (p(x_n, x_n, x_{n-1}), p(x_{n+1}, x_{n+1}, x_n), p(x_n, x_n, x_{n-1}))$$
$$= \alpha (P_{n-1}, Q_n, P_{n-1}),$$

which gives again by (ii) of Definition 1.13 that $Q_n \leq k_2 P_{n-1}$ for some $k_2 \in [0,1)$.

Therefore if $k = \max (k_1, k_2)$ then $P_n \leq k Q_{n-1}$ and $Q_n \leq k P_{n-1}$ so that $P_n \leq k^2 P_{n-2}$ and $Q_n \leq k^2 Q_{n-2}$; and hence $A_n \leq k^2 p(x_{n-1}, x_{n-1}, x_{n-2}) = k^2 A_{n-2}$ for $n \geq 1$. On repeated use of this inequality we find that

(2.5) $A_n \leq k^n p(x_1, x_1, x_0)$ or $k^n p(x_0, x_0 x_1)$ according as n is even or odd.

Since $0 \leq k < 1$, it follows from (2.5) that

(2.6) $\lim_{n\to\infty} A_n = 0$

Now ,by (a') of Definition 1.10,we have
(2.7) $p(x, x, y) \leq 2\, p(a, a, x) + p(a, a, y)$ for $x, y, a \in X$.
Therefore for any $m = n + r > n \geq 1$, we get by (2.7)

$$p(x_n, x_n, x_m) \leq 2\, p(x_{n+1}, x_{n+1}, x_n) + p(x_{n+1}, x_{n+1}, x_m)$$
$$= 2 A_n + p(x_{n+1}, x_{n+1}, x_m),$$

which as repeated use gives ,

$$p(x_n, x_n, x_m) \leq 2 A_n + p(x_{n+1}, x_{n+1}, x_m) \leq 2 A_n + 2 A_{n+1} + p(x_{n+2}, x_{n+2}, x_m) \leq \ldots \leq 2(A_n + A_{n+1} + \ .+A_{n+r-1}) + p(x_m, x_m, x_m)$$

That is , for $m = n + r > n \geq 1$,

(2.8) $p(x_n, x_n, x_m) \leq \alpha_n$ where $\alpha_n = 2(A_n + A_{n+1} + \ .+A_{n+r}) + p(x_m, x_m, x_m)$

In view of (2.6) and (2.4) we have $\alpha_n \to 0$ as $n \to \infty$. Therefore by (ii) of Lemma 1.18,(2.8) gives that $\{x_n\}$ is a Cauchy sequence in (X, S). Since (X, S) is complete there is a $z \in X$ such that $\{x_n\}$ converges to z. That is ,
(2.9) $\lim_{n\to\infty} S(x_n, x_n, z)=0$, proving the first part of the theorem.

To prove part (ii) suppose $z \in X$ is such that $p(z, z, z) = 0$.
First of all , a particular case of (2.9),(1.9) and $(2.1)''$ give

$$0 = \lim_{n\to\infty} S(x_{2n+1}, x_{2n+1}, z) = \lim_{n\to\infty} S(gx_{2n}, gx_{2n}, z)$$
$$= S(\lim_{n\to\infty} gx_{2n}, \lim_{n\to\infty} gx_{2n}, z)$$
$$= S(g(\lim_{n\to\infty} x_{2n}), g(\lim_{n\to\infty} x_{2n}), z)$$
$$= S(gz, gz, z)$$

from which it follows that $gz = z$.That is , z is a fixed point of g . By (2.2), we have

$p(fz, fz, z) = p(fz, fz, gz) \leq \alpha (p(z, z, z), p(fz, fz, z), p(gz, gz, z))$
$= \alpha (p(z, z, z), p(fz, fz, z), p(z, z, z))$
so that $p(fz, fz, z) \leq k\, p(z, z, z)$ for some $k \in [0,1)$;
and since $p(z, z, z) = 0$ it follows $p(fz, fz, z) = 0$.

Hence ,by $(2.1)'$, $S(fz, fz, z) = 0$ giving $fz = z$.
Thus z is a common fixed point of f and g.
To prove the uniqueness of z , assume $u \in X$ is such that $f u = g u = u$.
Again ,by (2.2) ,we get

$$P(u, u, u) = p(fu, fu, gu) \leq \alpha\big(p(u, u, u), p(fu, fu, u), p(gu, gu, u)\big)$$
$$= \alpha(p(u, u, u), p(u, u, u), p(u, u, u))$$

so that $p(u, u, u) \leq k\, p(u, u, u)$ for some $k \in [0,1)$ and this is possible only if $p(u, u, u) = 0$.
Now $p(u, u, z) = p(fu, fu, gz) \leq \alpha\big(p(u, u, z), p(fu, fu, u). p(gz, gz, z)\big)$

$$= \alpha\big(p(u, u, z), p(u, u, u), p(z, z, z)\big)$$
$$= \alpha(p(u, u, z),0,0)$$

and hence $p(u, u, z) \leq k .0 = 0$ giving $p(u, u, z) = 0$.

Thus if u and z are two common fixed points of f and g then $p(u, u, u) = p(u, u, z) = 0$ giving $u = z$,by (i) of Lemma 1.18 .
In the case $p(x, y, z) = S'(x, y, z)$ for $x, y, z \in X$ where S' is another S-metric on X, the conditions in (i) and (ii) of the theorem hold; and also in view of Remark 1.17 we can restate this theorem as follows :

2.10 Theorem .

Suppose (X, S) is a complete S-metric space and S' is another S-metric on X such that

$(2.10)'$ $S(x, x, y) \leq S'(x, x, y)$ for all $x, y \in X$.

If f and g are self maps of X such that

$(2.10)''$ g is continuous on(X, S)
and if there is an $\alpha \in A$ for which
$(2.10)'''$ $S'(fx, fx, gy) \leq \alpha\, (S'(x, y, z), S'(fx, fx, x), S'(gy, gy, y))$ for all $x, y \in X$.

Then for any $x_0 \in X$, the sequence $\{x_n\}$ defined for $n \geq 1$, by $x_n = fx_{n-1}$ or $g\, x_{n-1}$ according as n is even or odd, converges to $z \in X$ which is the common fixed point of f and g.

2.11 Corollary (Theorem 7 of [1]) Suppose (X, d) is a complete metric space and δ is another metric on X such that

$(2.11)'$ $d(x, y) \leq \delta(x, y)$ for all $x, y \in X$.

If f and g are self maps of X such that

$(2.11)''$ g is continuous on(X, d)
and there is an $\alpha \in A$ such that
$(2.11)'''$ $\delta(fx, gy) \leq \alpha\, (\delta(x, y), \delta(fx, x), \delta(gy, y))$ for all $x, y \in X$.

Then to each $x_0 \in X$, the sequence $\{x_n\}$ defined for $n \geq 1$, by $x_n = fx_{n-1}$ or gx_{n-1} according as n is even or odd, converges to the unique common fixed point z of f and g.

Proof :

Define S and S' on X^3 by $S(x, y, z) = d(x, z) + d(y, z)$ and $S'(x, y, z) = \delta(x, z) + \delta\, (y, z)$ for all $x, y , z \in X$ .Then (X, S) is a S-metric space and S' is another S-metric on X. Also since $S(x, x, y) = 2\, d(x, y)$ and $S'(x, x, y) = 2\, \delta(x, y)$, it follows$(2.11)'$ implies$(2.10)'$; $(2.11)''$ gives $(2.10)''$ and $(2.11)'''$is equivalent to$(2.10)'''$ in the S-metric space (X, S) . Therefore ,by

Theorem 2.10 ,z is the unique common fixed point of f and g.

Now we prove the following

2.12 Theorem
Suppose $(X.S)$ is a complete S –metric space and p is a weak S-metric on it such that $p(x, x, y) = p(y, y, x)$ for all $x, y \in X$ and $p(x, x, .): X \to [0, \infty)$ is continuous on X for each $x \in X$.

If $\{f_n\}$ is a sequence of self maps of X for which there is an $\alpha \in A$ such that

$\big(f_i , f_j\big) \in \zeta_\alpha{}^p$ *for any* $i \neq j$ then

(i) for any $x_0 \in X$ the sequence $\{x_n\}_{n=1}^{\infty}$ defined by $x_n = fx_{n-1}$ converges to a point $z \in X$, provided $\lim_{n\to\infty} p(x_n, x_n, x_n) = 0$

and also

(ii) any such $z \in X$ with $p(z, z, z) = 0$ is the unique common fixed point of the sequence $\{f_n\}$.

Proof: By the hypothesis, for any $i \neq j$,

(2.13) $p((f_i x, f_i x, f_j y) \leq \alpha(p(x, x, y), p(f_i x, f_i x, x), p(f_j y, f_j y, y))$ for all $x, y \in X$.

Let $x_0 \in X$ and let $x_n = f_n x_{n-1}$ for $n \geq 1$ be such that
(2.14) $\lim_{n\to\infty} p(x_n, x_n, x_n) = 0$.
Write $A_n = p(x_{n+1}, x_{n+1}, x_n)$ for $n \geq 0$.
Then
$$\begin{aligned} A_n &= p(f_{n+1}x_n, f_{n+1}x_n, f_n x_{n-1}) \\ &\leq \alpha(p(x_n, x_n, x_{n-1}), p(f_{n+1}x_n, f_{n+1}x_n, x_n), p(f_n x_{n-1}, f_n x_{n-1}, x_{n-1}) \\ &= \alpha(p(x_n, x_n, x_{n-1}), p(x_{n+1}, x_{n+1}, x_n), p(x_n, x_n, x_{n-1})) \\ &= \alpha (A_{n-1}, A_n, A_{n-1}), \end{aligned}$$

so that by (ii) of Definition 1.13, $A_n \leq k A_{n-1}$ for some $k \in [0,1)$, which on repeated use gives

(2.15) $A_n \leq k A_{n-1} \leq k^2 A_{n-2} \leq \qquad \leq k^n A_0$
and since $0 \leq k < 1$, (2.15) gives
(2.16) $\lim_{n\to\infty} A_n = 0$.

Now if $m = n + r > n \geq 1$ then as in the proof of Theorem 2.1 we have

$p(x_n, x_n, x_m) \leq 2(A_n + A_{n+1} + \quad . + A_{n+r-1}) + p(x_m, x_m, x_m) = \alpha_n$ and
since $\alpha_n \to 0$ as $n \to \infty$ (in view of (2.13) and (2.16)), it follows from (ii) of Lemma 1.18, that $\{x_n\}$ is a Cauchy sequence in (X, S). Since (X, S) is complete there is a $z \in X$ such that $\lim_{n\to\infty} x_n = z$ in (X, S), proving part (i) of the theorem.
To prove part (ii) suppose $z \in X$ is such that $p(z, z, z) = 0$. Fix a natural number n. By(2.7), we have for $m > n$,
(2.17)
$$\begin{aligned} p((z, z, f_n z) &\leq 2\, p(x_{m+1}, x_{m+1}, z) + p(x_{m+1}, x_{m+1}, f_n z)) \\ &= 2\, p(x_{m+1}, x_{m+1}, z) + p(f_{m+1}x_m, f_{m+1}x_m, f_n z) \\ &\leq 2\, p(x_{m+1}, x_{m+1}, z) + \alpha\, (p(x_m, x_m, z), p(x_{m+1}, x_{m+1}, x_m), p(f_n z, f_n z, z), \end{aligned}$$

since $p(x, x, y) = p(y, y, x)$ for all $x, y \in X$, by the hypothesis.
Now letting m to ∞, using the continuity of α as R_+^3 and that of $p(z, z, .)$ on X; and (2.16) in (2.17) we get

$$\begin{aligned} p(z, z, f_n z) &\leq 2\, p\, (z, z, z) + \alpha\, (p(z, z, z), 0, (p(z, z, f_n z)) \\ &= \alpha\, (0, 0, p(z, z, f_n z) \end{aligned}$$

from which it follows $p(z, z, f_n z) \leq k.0 = 0$ so that $p(z, z, f_n z) = 0$.
Thus if $z \in X$ is such that $p(z, z, z) = 0$ then $p(z, z, f_n z) = 0$.
That is, $p(z, z, f_n z) = p(z, z, z) = 0$ giving $f_n z = z$. Since n is arbitrary z is a common fixed point of the sequence $\{f_n\}$.
For uniqueness of z, let $u \in X$ be such that $f_n u = u$ for $n \geq 1$. Then for any $i \neq j$, we have
$$\begin{aligned} p(u, u, u) = p(f_i u, f_i u, f_j u) &\leq \alpha(p(u, u, u), p(f_i u, f_i u, u), p(f_j u, f_j u, u)) \\ &= \alpha\, (p(u, u, u), p(u, u, u), p(u, u, u)) \end{aligned}$$
so that $p(u, u, u) \leq k\, p(u, u, u)$ for some $k \in [0,1)$ and this is possible only if $p(u, u, u) = 0$.

Also

$$\begin{aligned} p(u, u, z) = p(f_i u, f_i u, f_j z) &\leq \alpha(p(u, u, z), p(f_i u, f_i u, u), p(f_j z, f_j z, z)) \\ &= \alpha\, (p(u, u, z), p(u, u, u), p(z, z, z)) \\ &= \alpha\, (p(u, u, z), 0, 0) \end{aligned}$$
gives $p(u, u, z) \leq k.\, 0 = 0$ and hence $p(u, u, z) = 0$.

Thus $p(u, u, z) = p(u, u, u) = 0$ so that $z = u$, by (i) of Lemma 1.18, completing the proof of the theorem.
In the case $p = S$, the S-metric on X, the conditions in (i) and (ii) of the theorem as well as the conditions in the hypothesis are true so that the theorem can be restated in this particular case as follows:

2.18 Theorem

Suppose (X, S) is a complete S-metric space and $\{f_n\}$ is a sequence of self maps of X such that there is an $\alpha \in A$ for which (2.18)' $S(f_i x, f_i x, f_j y) \leq \alpha(S(x,x,y), S(f_i x, f_i x, x), S(f_j y, f_j y, y))$ holds for all $x, y \in X$ and for all $i \neq j$. Then for any $x_0 \in X$ the sequence $\{x_n\}$ defined by $x_n = f_n x_{n-1}$ for $n \geq 1$ converges to a point $z \in X$ and this z is the unique common fixed point of the sequence $\{f_n\}$.

2.19. Corollary ([1],Theorem 6) Suppose (X, d) is a complete metric space and $\{f_n\}$ is a sequence of self maps on X such that there is an $\alpha \in A$ for which

(2.19)' $d(f_i x, f_j y) \leq \alpha(d(x,y), d(f_i x, x), d(f_j y, y))$ for all $x, y \in X$ holds for all $i \neq j$. Then for any $x_0 \in X$ the sequence $\{x_n\}$ defined by $x_n = f_n x_{n-1}$ for $n \geq 1$ converges to a point $z \in X$ and which is the common fixed point for the sequence $\{f_n\}$.
Proof: On the lines similar to corollary 2.11.

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