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ASIAN JOURNAL OF SCIENCE AND TECHNOLOGY

ISSN: 0976-3376

Asian Journal of Science and Technology
Vol. 07, Issue, 05, pp.2960-2970, May, 2016

RESEARCH ARTICLE

BETA DISTRIBUTION AND INFERENCES ABOUT THE BETA FUNCTIONS

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ARTICLE INFO

Article History:
Received 19th February, 2016
Received in revised form
21st March, 2016
Accepted 19th April, 2016
Published online 30th May, 2016

Key words:

Beta function;
Incomplete Beta distributions;
Hypergeometric function.

ABSTRACT

In this study, general information about the Beta function and Beta distributions were given. Usage areas of Beta distributions are specified. Moments were expressed in general. The distribution of the joint probability density function of two independent random Gamma variables and the distribution of a random sample range were found to be Beta distribution. The relations were investigated between the incomplete Beta distribution and other distributions. Incomplete Beta function could be expressed as the term of hypergeometric functions. As a result, some inferences about Beta distribution were found.

INTRODUCTION

The Beta distribution is given by

$$f(x;\alpha,\beta)=\frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1} \qquad \text{.......................... (1)}$$

where the parameters p and q are positive real quantities and the variable x satisfies $0\le x\le 1$. The quantity $B(\alpha,\beta)$ is the Beta function defined in terms of the more common Gamma function as

$$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad \text{...........................(2)}$$

for $\alpha=\beta=1$ the Beta distribution simply becomes a uniform distribution between zero and one. For $\alpha=1$ and $\beta=2$ or vice versa we get triangular shaped distributions, $f(x)=2-2x$ and $f(x)=2x$. For $\alpha=\beta=2$, a distribution of parabolic shape, $f(x)=6x(1-x)$ is obtained. More generally, if p and q both are greater than one, the distribution has a unique mode at $x=(\alpha-1)/(\alpha+\beta-2)$ and is zero at the end-points. If α and/or β is less than one $f(0)\to\infty$ and/or $f(1)\to\infty$ and the distribution is said to be J-shaped. In Figure 1 below we show the Beta distribution for two cases: $\alpha=\beta=2$ and $\alpha=6$, $\beta=3$ was shown (Walck, 2007). The Beta distribution and Beta function are widely used in mathematics and statistics. Some of these uses are given below. The Beta distribution is used in the calculation of the expected duration of the activities in Project Evaluation and Review Technique. In addition, with the help Project Evaluation and Review Technique, by using the formula $Z=\frac{t-t_e}{\sigma}$ and the table of standard normal distribution, it is possible to calculate the probability of completion of the project at different times.

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Here Z is the likelihood of completion of the Project with a certain period of time, t is the programmed duration of the project, t_e is the expected time of completion the project, σ is the standard deviation (Yamak, 1994). In stochastic techniques, which are used more than once in estimate of the time of activity in project management, three estimates are made for each activity. of the estimate of the duration in project management. These are optimistic duration, possible duration and pessimistic time.

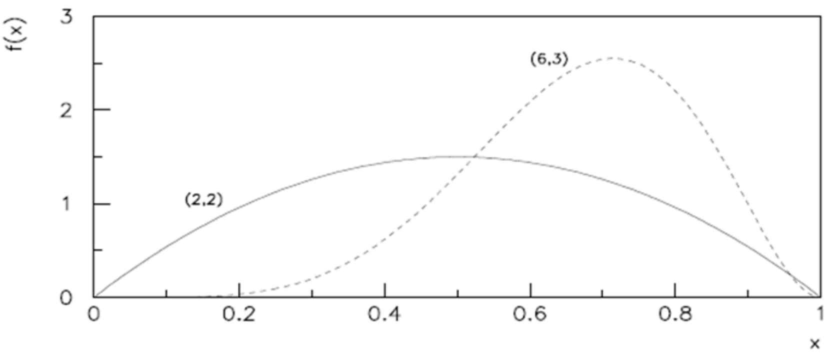

Figure 1. Examples of Beta distributions (for two cases $\alpha = \beta = 2$ and $\alpha = 6$, $\beta = 3$)

The estimate of these three times is assumed to agree with the Beta probability distribution. The expected duration of the activity is estimated by using the following formula.

$$t_e = \frac{t_o + 4(t_m) + t_p}{6}$$

According to Beta distribution, the probability of the calculated time being shorter or longer than the expected time is 50%. In Beta distribution, variance, σ^2, showing the deviation from the average, is calculated as follows (Gido and Clements, 1999).

$$\sigma^2 = \left(\frac{t_p - t_o}{6} \right)^2$$

where t_e is the expected period of time, t_o is optimistic time, t_m is possible period of time, t_p is pessimistic time (Ozturk, 2005).

The purpose of this study is to define the Beta function and the Beta distribution, to explore their features, to make important inferences and to determine their relations with the other distributions and functions used in mathematics and statistics.

MATERIALS AND METHODS

Definition and Properties of Beta Function

Beta function $B(\alpha, \beta)$ is defined as the following formulas

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$$ (Spiegel, 1974).

The beta function is related to the gamma function through the equation (3).

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Equation (3) is very useful in dealing with the beta function allowing us to take advantage of the properties of the gamma function (Casella and Berger, 2002). Take $\alpha = \beta = \frac{1}{2}$ in Equation (3). If the transformation $x = \sin^2\theta$ is done, $dx = 2\sin\theta\cos\theta d\theta$ is obtained. For the integral limits, the following transformation was done:

$x = 0 \Rightarrow \theta = 0$ and $x = 1 \Rightarrow \theta = \frac{\pi}{2}$

In this case, the integral of the function was found by using the following operation.

$$\int_0^{\pi/2} (\sin^2\theta)^{\frac{1}{2}-1}(1-\sin^2\theta)^{\frac{1}{2}-1} 2\sin\theta\cos\theta d\theta = \int_0^{\pi/2} (\sin^2\theta)^{-\frac{1}{2}}(1-\sin^2\theta)^{-\frac{1}{2}} 2\sin\theta\cos\theta d\theta$$

$$= \int_0^{\pi/2} \sin^{-1}\theta\cos^{-1}\theta\, 2\sin\theta\cos\theta d\theta = 2(\theta\Big|_0^{\pi/2}) = 2\left(\frac{\pi}{2}-0\right) = \pi$$

So the following equation was obtained:

$$\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \pi \Rightarrow \frac{\Gamma^2\left(\frac{1}{2}\right)}{\Gamma(1)} = \pi$$

and

$$\Gamma^2\left(\frac{1}{2}\right) = \pi \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \qquad \text{……………………….(4)}$$

Some of the integrals given in in terms of sine and cosine can be calculated by taking advantage of the Beta function. For example, any $\alpha, \beta \in$ □ for the based on integral, the integral is given

$$I = \int_0^{\pi/2} (\sin u)^{\alpha}(\cos u)^{\beta} du$$

where $\alpha = 2x-1$, $\beta = 2y-1$

from here the following integral was found as

$$I = \int_0^{\pi/2} (\sin^2 u)^{x-1}\sin u(\cos^2 u)^{y-1}\cos u\, du$$

By using the transformation ($\sin^2 u = \theta, 2\sin u\cos u = d\theta$) the following integral were found as the form of Beta function.

$$I = \int_0^1 \theta^{x-1}(1-\theta)^{y-1}\frac{d\theta}{2} = \frac{1}{2}B(x,y) = \frac{1}{2}B\left(\frac{\alpha+1}{2},\frac{\beta+1}{2}\right)$$

Moments of Beta Distributions

Moments of Beta distributions, generally algebraic moments are given in terms of the Beta function by

$$\mu_n = \frac{B(\alpha+n,\beta)}{B(\alpha,\beta)} = \frac{\Gamma(\alpha+n)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+n)\Gamma(\alpha)} \quad \text{(Walck, 2007).}$$

$$\mu_n = E(X^n) = \int_0^1 x^n f(x)dx = \int_0^1 x^n \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}dx$$

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 x^{n+\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}B(\alpha+n,\beta)$$

$$= \frac{B(\alpha+n,\beta)}{B(\alpha,\beta)} = \frac{\Gamma(\alpha+n)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+n)\Gamma(\alpha)}$$

for n=1, $\mu_1 = E(X) = \frac{\alpha}{\alpha+\beta}$

for n=2, $\mu_2 = E(X^2) = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$

and variance

$$Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \quad \text{(Öztürk, 2005).}$$

for n=3, $\mu_3 = E(X^3) = \dfrac{2\alpha\beta(\beta-\alpha)}{(\alpha+\beta)^3(\alpha+\beta+1)(\alpha+\beta+2)}$

for n=4, $\mu_4 = E(X^4) = \dfrac{3\alpha\beta(2(\alpha+\beta)^2+\alpha\beta(\alpha+\beta-6))}{(\alpha+\beta)^4(\alpha+\beta+1)(\alpha+\beta+2)(\alpha+\beta+3)}$ (Walck, 2007).

Example 1. If X and Y are independent gamma random variables with parameters (α,λ) and (β,λ) respectively, compute the joint density of $U = X+Y$ and $V = \frac{X}{X+Y}$

Solution 1. The joint density of X and Y is given by

$$f_{X,Y}(x,y) = \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}\frac{\lambda e^{-\lambda y}(\lambda y)^{\beta-1}}{\Gamma(\beta)} = \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)}e^{-\lambda(x+y)}x^{\alpha-1}y^{\beta-1}$$

Now, if $g_1(x,y) = x+y$, $g_2(x,y) = \frac{x}{x+y}$ then

$\frac{\partial g_1}{\partial x} = \frac{\partial g_1}{\partial y} = 1$, $\frac{\partial g_2}{\partial x} = \frac{y}{(x+y)^2}$, $\frac{\partial g_2}{\partial y} = -\frac{x}{(x+y)^2}$

and so

$$J(x,y) = \begin{vmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{y}{(x+y)^2} & -\frac{x}{(x+y)^2} \end{vmatrix} = -\frac{1}{x+y}$$

Finally, as the equations $u = x+y$ and $v = \frac{x}{x+y}$ have as their solutions $x = uv$, $y = u(1-v)$,

$$f_{U,V}(u,v) = f_{U,V}(uv, u(1-v))u = \frac{\lambda e^{-\lambda u}(\lambda u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\frac{v^{\alpha-1}(1-v)^{\beta-1}\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

is found. Hence $X+Y$ and $X/(X+Y)$ are independent, with $X+Y$ having a gamma distribution with parameters $(\alpha+\beta,\gamma)$ and $X/(X+Y)$ having

$$f_V(v) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}v^{\alpha-1}(1-v)^{\beta-1}, \quad 0<v<1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

This is called the Beta density with parameters (α,β) (Akdi, 2005).

Example 2. Distribution of the range of a random sample

Suppose that n independent and identically distributed random variables $X_1, X_2, \ldots, X_n$ are observed. The random variable R, defined by $R = X_{(n)} - X_{(1)}$, is called the range of the observed random variables. If the random variables, X_i have distribution function F and density function f, then the distribution of R can be obtained from Equation (6) as follows: for $a \geq 0$

$$f_{X_{(i)},X_{(j)}}(x_i,x_j) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\left[F(X_i)\right]^{i-1}\left[F(X_j)-F(X_i)\right]^{j-i-1}$$

$$\left[1-F(X_j)\right]^{n-j}f(x_i)f(x_j), \text{ for } \forall\ X_i < X_j \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

$$P(R \leq a) = P\left\{X_{(n)} - X_{(1)} \leq a\right\} = \int\int_{X_n - X_1 \leq a} f_{X_{(1)},X_{(n)}}(x_1,x_n)dx_1dx_n$$

$$= \int_{-\infty}^{\infty} \int_{x_1}^{x_1+a} \frac{n!}{(n-2)!} \left[F(X_n) - F(X_1)\right]^{n-2} f(x_1, x_n) dx_n dx_1$$

Making the change of variable $y = F(X_n) - F(X_1)$, $dy = f(x_n)dx_n$ yields

$$\int_{x_1}^{x_1+a} \left[F(X_n) - F(X_1)\right]^{n-2} f(x_n)dx_n = \int_{0}^{F(X_1+a)-F(X_1)} y^{n-2}dy = \frac{1}{n-1}\left[F(X_1+a) - F(X_1)\right]^{n-1}$$

and thus the Equation (7) is found.

$$P\left\{R \le a\right\} = n\int_{-\infty}^{\infty} \left[(F(X_1)+a) - F(X_1)\right]^{n-1} f(x_1)dx_1 \qquad \text{……………………..(7)}$$

Equation (7) can be explicitly evaluated only in a few special cases. One such case is when the X_i's are all uniformly distributed on (0,1). In this case, the following equation could be obtained from (7) for $0 < a < 1$,

$$P\left\{R \le a\right\} = n\int_{0}^{1} \left[(F(X_1)+a) - F(X_1)\right]^{n-1} f(x_1)dx_1$$

$$= n\int_{0}^{1} a^{n-1}dx_1 + n\int_{1-a}^{1} (1-x_1)^{n-1}dx_1 = n(1-a)a^{n-1} + a^n$$

Differentiation yields that the density function of the range is given, in this case, by

$$f_R(a) = \begin{cases} n(n-1)a^{n-2}(1-a) & , \quad 0 \le a \le 1 \\ 0 & , \quad otherwise \end{cases}$$

That is, the range of n independent Uniform (0,1) random variables is a Beta variable with parameters n-1, 2 (Ross, 2000).

Example 3. Uniform order statistics probability density function

Let us consider the standard uniform distribution with density function $f(x) = 1$, $0 \le x \le 1$ and cumulative distribution function $F(x) = x$, $0 \le x \le 1$. Then,

$$F_{X_{(i)}}(x) = \sum_{r=i}^{n} \binom{n}{r} x^r (1-x)^{n-r} = \int_0^t \frac{n!}{(i-1)!(n-i)!} x^{i-1}(1-x)^{n-i}\, dx, \qquad 0 \le x \le 1$$

The density function of $X_{(i)} (1 \le i \le n)$ is found as

$$f_{X_{(i)}}(x) = \frac{n!}{(i-1)!(n-i)!} x^{i-1}(1-x)^{n-i}, \qquad 0 \le x \le 1 \qquad \text{…………………..(8)}$$

for $x \in (0,1)$. Hence $X_{(i)} \sim Beta(i, n-i+1)$ (Arnold et. al., 2008). From this we can deduce that

$$E(X_{(i)}) = \frac{i}{n+1}$$

and

$$Var(X_{(i)}) = \frac{i(n-i+1)}{(n+1)^2(n+2)}$$

The Incomplete Beta Function

The incomplete Beta function is defined as

$$I_x(\alpha, \beta) = \frac{B_x(\alpha, \beta)}{B(\alpha, \beta)} = \frac{1}{B(\alpha, \beta)} \int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt \qquad \text{………………………(9)}$$

where $\alpha, \beta > 0$ and $0 \le x \le 1$.

The function $B_x(\alpha, \beta)$, often also called the incomplete Beta function, satisfies the following formula

$$B(\alpha,\beta) = \int_0^{\frac{x}{1-x}} \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du = B_1(\beta,\alpha) - B_{1-x}(\beta,\alpha)$$

$$= x^{\alpha}\left[\frac{1}{\alpha} + \frac{1-\beta}{\alpha+1} + \frac{(1-\beta)(2-\beta)}{2!(\alpha+2)}x^2 + ... + \frac{(1-\beta)(2-\beta)...(n-\beta)}{n!(\alpha+n)}x^n + ...\right]$$

In Figure 2, the incomplete Beta function is shown for a few (α, β) values. Note that by symmetry the (1, 5) and (5, 1) curves are reflected around the diagonal. For large values of α and β the curve rises sharply from near zero to near one around $x = \alpha/(\alpha+\beta)$.

In order to obtain $I_x(\alpha, \beta)$ the series expansion

$$I_x(\alpha,\beta) = \frac{x^{\alpha}(1-x)^{\beta}}{\alpha B(\alpha,\beta)}\left[1 + \sum_{n=0}^{\infty} \frac{B(\alpha+1, n+1)}{B(\alpha+\beta, n+1)} x^{n+1}\right]$$

is not the most useful formula for computations. The continued fraction formula

$$I_x(\alpha,\beta) = \frac{x^{\alpha}(1-x)^{\beta}}{\alpha B(\alpha,\beta)}\left[\frac{1}{1+}\frac{d_1}{1+}\frac{d_2}{1+}...\right]$$

turns out to be a better choice (Press et al., 1992). Here

$$d_{2m+1} = -\frac{(\alpha+m)(\alpha+\beta+m)x}{(\alpha+2m)(\alpha+2m+1)} \quad \text{and} \quad d_{2m} = \frac{m(\beta-m)x}{(\alpha+2m-1)(\alpha+2m)}$$

and the formula converges rapidly for $x < \frac{(\alpha+1)}{(\alpha+\beta+1)}$. For other x-values the same formula may be used after applying the symmetry relation

$$I_x(\alpha,\beta) = 1 - I_{1-x}(\beta,\alpha)$$

$I_x(\alpha,\beta)$

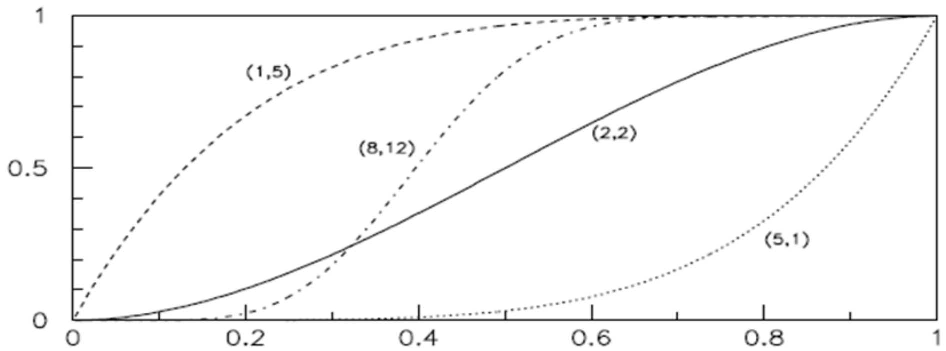

Figure 2. The incomplete Beta function (for four cases: $\alpha = 1$ and $\beta = 5$, $\alpha = 8$ and $\beta = 12$, $\alpha = \beta = 2$ and $\alpha = 5, \beta = 1$)

For higher values of α and β, well already from $\alpha + \beta > 6$, the incomplete Beta function may be approximated by.

For $(\alpha+\beta+1)(1-x) \le 0.8$ using an approximation to the chi-square distribution in the variable $\chi^2 = (\alpha+\beta-1)(1-x)(3-x) - (1-x)(\beta-1)$ with $n = 2\beta$ degrees of freedom.

For $(\alpha+\beta+1)(1-x) \geq 0,8$ using an approximation to the standard normal distribution in the variable

$$z = \frac{3\left[w_1\left(1-\frac{1}{9\beta}\right) - w_2\left(1-\frac{1}{9\alpha}\right)\right]}{\sqrt{\frac{w_1^2}{\beta} + \frac{w_2^2}{\alpha}}}$$

where $w_1 = \sqrt[3]{\beta x}$ and $w_2 = \sqrt[3]{\alpha(1-x)}$.

In both cases the maximum difference to the true cumulative distribution is below 0.005 all way down to the limit where $\alpha + \beta = 6$ (Abramowitz and Stegun, 1965).

Relations the Between I_x the Incomplete Beta function and some distributions

The incomplete Beta function I_x is connected to the binomial distribution for integer values of α by

$$1 - I_x(\alpha,\beta) = I_{1-x}(\beta,\alpha) = (1-x)^{\alpha+\beta-1} \sum_{i=0}^{\alpha-1} \binom{\alpha+\beta-1}{i} \left(\frac{x}{1-x}\right)^i$$

or expressed in the opposite direction

$$\sum_{s=\alpha}^{n} \binom{n}{s} p^s (1-p)^{n-s} = I_p(\alpha, n-\alpha+1)$$

The negative binomial distribution with parameters n and p is related to the incomplete Beta function via the relation

$$\sum_{s=\alpha}^{n} \binom{n+s-1}{s} p^n (1-p)^s = I_{1-p}(\alpha, n)$$

The symmetric integral of the t-distribution with n degrees of freedom, often denoted $A(t/n)$, is given by

$$A(t/n) = \frac{1}{\sqrt{n}B\left(\frac{1}{2},\frac{n}{2}\right)} \int_{-t}^{t} \left(1+\frac{x^2}{n}\right)^{-\frac{n+1}{2}} dx = \frac{2}{\sqrt{n}B\left(\frac{1}{2},\frac{n}{2}\right)} \int_{0}^{t} \left(\frac{n}{n+x^2}\right)^{\frac{n+1}{2}} dx$$

$$= \frac{2}{\sqrt{n}B\left(\frac{1}{2},\frac{n}{2}\right)} \int_{0}^{\frac{t^2}{n+t^2}} (1-y)^{\frac{n+1}{2}} \frac{n}{2}\left(\frac{1}{1-y}\right)^2 \sqrt{\frac{1-y}{ny}} dy$$

$$= \frac{1}{B\left(\frac{1}{2},\frac{n}{2}\right)} \int_{0}^{\frac{t^2}{n+t^2}} y^{-\frac{1}{2}} (1-y)^{\frac{n}{2}-1} dy = \frac{B_z\left(\frac{1}{2},\frac{n}{2}\right)}{B\left(\frac{1}{2},\frac{n}{2}\right)} = I_z\left(\frac{1}{2},\frac{n}{2}\right)$$

with $z = t^2/(n+t^2)$.

The cumulative F-distribution with m and n degrees of freedom is given by

$$F(x) = \frac{1}{B\left(\frac{m}{2},\frac{n}{2}\right)} \int_{0}^{x} \frac{m^{\frac{m}{2}} n^{\frac{n}{2}} F^{\frac{m}{2}-1}}{(mF+n)^{\frac{m+n}{2}}} dF = \frac{1}{B\left(\frac{m}{2},\frac{n}{2}\right)} \int_{0}^{x} \left(\frac{mF}{(mF+n)}\right)^{\frac{m}{2}} \left(\frac{n}{mF+n}\right)^{\frac{n}{2}} \frac{dF}{F}$$

$$= \frac{1}{B\left(\frac{m}{2},\frac{n}{2}\right)} \int_0^{\frac{mx}{mx+n}} y^{\frac{m}{2}} (1-y)^{\frac{n}{2}} \frac{dy}{y(1-y)} = \frac{1}{B\left(\frac{m}{2},\frac{n}{2}\right)} \int_0^{\frac{mx}{mx+n}} y^{\frac{m}{2}-1} (1-y)^{\frac{n}{2}-1} dy$$

$$= \frac{B_z\left(\frac{m}{2},\frac{n}{2}\right)}{B\left(\frac{m}{2},\frac{n}{2}\right)} = I_z\left(\frac{m}{2},\frac{n}{2}\right)$$

with $z = mx/(n+mx)$. Here we have made the substitution $y = mF/(mF+n)$, leading to $dF/F = dy/y(1-y)$, in simplifying the integral (Walck, 2007).

Example 4. The joint cumulative distribution function of $X_{(i)}$ and $X_{(j)}$ $(1 \le i \le j \le n)$ is the tail probability [over the rectangular region $(j,i),(j,i+1),...,(n,n)$] of a bivariate binomial distribution. By using the identity that

$$\sum_{s=j}^{n}\sum_{r=i}^{s} \frac{n!}{r!(s-r)!(n-s)!} p_1^r (p_2 - p_1)^{s-r} (1-p_2)^{n-s}$$

$$= \int_0^{p_1}\int_{t_1}^{p_2} \frac{n!}{(i-1)!(j-i-1)!(n-j)!} t_1^{i-1} (t_2 - t_1)^{j-i-1} (1-t_2)^{n-j} dt_2 dt_1 ,\ 0 < p_1 < p_2 < 1 \quad \text{.....................(10)}$$

the joint cumulative distribution function of $X_{(i)}$ and $X_{(j)}$ could be equivalently written as

$$F_{x_{(i)},x_{(j)}}(x_i, x_j) = \int_0^{F(x_i)}\int_{t_1}^{F(x_j)} \frac{n!}{(i-1)!(j-i-1)!(n-j)!}$$

$$\times t_1^{i-1} (t_2 - t_1)^{j-i-1} (1-t_2)^{n-j} dt_2 dt_1 ,\ -\infty < x_i < x_j < \infty \quad \text{...........................(11)}$$

which may be noted to be an incomplete bivariate Beta function (Arnold et al., 2008).

Table 1 summarizes the relations between the cumulative, distribution, functions of some standard probability density functions and the incomplete Gamma and Beta functions (Walck, 2007).

Table 1. The incomplete Beta distribution with other distribution

Distributions	Parameters	Cumulative distribution	Range
Beta	α , β	$F(x) = I_x(\alpha, \beta)$	$0 \le x \le 1$
Binom	n, p	$P(k) = I_{1-p}(n-k, k+1)$	k=0,1,2,…,n
F	m, n	$F(x) = I_{\frac{mx}{n+mx}}\left(\frac{m}{2},\frac{n}{2}\right)$	$x \ge 0$
Geometric	p	$P(k) = I_p(1,k)$	k=1,2,…
Negative Binom	n, p	$P(k) = I_p(n, k+1)$	k=0,1,2,…
Student t	n	$F(x) = \frac{1}{2} - \frac{1}{2} I_{\frac{x^2}{n+x^2}}\left(\frac{1}{2},\frac{n}{2}\right)$	$-\infty < x < 0$
		$F(x) = \frac{1}{2} + \frac{1}{2} I_{\frac{x^2}{n+x^2}}\left(\frac{1}{2},\frac{n}{2}\right)$	$0 \le x < \infty$

Hypergeometric Series and Hypergeometric Function

The hypergeometric function, sometimes called Gauss's differential equation, is given by (Arfken, 1970; Altın, 2011).

$$x(1-x)\frac{\partial^2 f(x)}{\partial x^2}+[\gamma-(\alpha+\beta+1)]\frac{\partial f(x)}{\partial x}-\alpha\beta f(x)=0$$

One solution is

$$f(x)=F(\alpha,\beta,\gamma;x)=1+\frac{\alpha\beta}{\gamma}\frac{x}{1!}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!}+\ldots \qquad \gamma\neq 0,-1,-2,\ldots \quad (12)$$

where α,β and γ are real or complex constants

The range of convergence is |x|<1 and x=1, for $\gamma>\alpha+\beta$, and x= 1, for $\gamma>\alpha+\beta-1$.

Pochmammer symbol

The expression of the $(\alpha)_r$ is defined as

$$(\alpha)_r=\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+r-1) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(13)$$

where α is a real or complex number, and r is zero or a positive number.

The expression determined as Pochmammer symbol has the following characteristics.

$$(\alpha)_r=\frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(14)$$

$$(\alpha)_{r+1}=\alpha(\alpha+1)_r \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(15)$$

Due to expression properties of Gamma function, the following expression is found.

$$\begin{aligned}\Gamma(\alpha+r)&=(\alpha+r-1)\Gamma(\alpha+r-1)=(\alpha+r-1)(\alpha+r-2)\Gamma(\alpha+r-2)\\&=\ldots\\&=(\alpha+r-1)(\alpha+r-2)\ldots(\alpha+1)\alpha\Gamma(\alpha)=(\alpha)_r\Gamma(\alpha)\end{aligned}$$

After that, the Equations (14) and (15) is divided by $\Gamma(\alpha)$ for both sides of the equations, the Equation (15) is converted the following Equation (16).

$$(\alpha)_{r+1}=\frac{\Gamma(\alpha+r+1)}{\Gamma(\alpha)}=\frac{\alpha\Gamma(\alpha+r+1)}{\alpha\Gamma(\alpha)}=\alpha\frac{\Gamma((\alpha+1)+r)}{\Gamma(\alpha+1)}=\alpha(\alpha+1)_r \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(16)$$

If the private Equation (14) at r = 0 is taken, then $(\alpha)_0=1$. Equation (12) can be written as hypergeometric series representation (13) taking into account the indication, respectively. By using the Equation (17),

$$F(\alpha,\beta,\gamma;x)=\sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r}{(\gamma)_r}\frac{x^r}{r!} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(17)$$

the generalized version of Equation (14) is found as the following expression.

$$F(\alpha_1,\alpha_2,\ldots,\alpha_p;\gamma_1,\gamma_2,\ldots,\gamma_q;x)=\sum_{r=0}^{\infty}\frac{(\alpha_1)_r(\alpha_2)_r\ldots(\alpha_p)_r}{(\gamma_1)_r(\gamma_2)_r\ldots(\gamma_q)_r}\frac{x^r}{r!}$$

Equation (17) based on the hypergeometric function α and β is symmetric. So $F(\alpha,\beta,\gamma;x)=F(\beta,\alpha,\gamma;x)$ is provided (Abramowitz et al., 1965).

An integral formula

One of the hypergeometric functions of correlation provided with an integral representation of this function. Since the definition of Beta function

$$B(u,v)=\int_0^1 x^{u-1}(1-x)^{v-1}\,dx$$

and properties of Pochammer symbol,

$$\frac{(B)_r}{(\gamma)_r} = \frac{B(\beta + r, \gamma - \beta)}{B(\beta, \gamma - \beta)} = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 t^{\beta + r - 1} (1-t)^{\gamma - \beta - 1} dt$$

could be written. From there,

$$F(\alpha, \beta, \gamma; x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} x^r \frac{(\beta)_r}{(\gamma)_r} = \frac{1}{B(\beta, \gamma - \beta)} \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} x^r \int_0^1 t^{\beta + r - 1} (1-t)^{\gamma - \beta - 1} dt$$

could be found and when symbol of sum is replaced with symbol of integration in this expression,

$$F(\alpha, \beta, \gamma; x) = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 t^{\beta - 1} (1-t)^{\gamma - \beta - 1} \left\{ \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} (xt)^r \right\} dt$$

could be found. On the other hand, due to the opening binomial expansion of $(1 - xt)^{-\alpha}$ and using

$$\sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} (xt)^r = (1 - xt)^{-\alpha}$$

the following expression is obtained.

$$F(\alpha, \beta, \gamma; x) = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 t^{\beta - 1} (1-t)^{\gamma - \beta - 1} (1 - xt)^{-\alpha} dt$$

where $|x| < 1$ and $\alpha > \beta > 0$ (Abramowitz et al ., 1965).

Also

$$B_x(\alpha, \beta) = \frac{x^{\alpha}}{\alpha} F(\alpha, 1 - \beta, \alpha + 1; x) \quad \text{…………………………..(18)}$$

can be expressed as an incomplete Beta function in the form of hypergeometric functions (Arfken, 1970).

RESULT AND DISCUSSION

With parameter (α, β), Beta function is expressed with the help of determined properties of Gamma distribution as if defined in the Equation (2). For $\alpha = \beta = 1$, the Beta distribution simply becomes a uniform distribution between zero and one. For $\alpha = 1$ and $\beta = 2$ or vice versa it converts triangular shaped distributions. It is easily obtained significant results by using the relationship between Gamma and Beta distributions. As an important result, the value of $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ is calculated by using the expression in Equation (4). This result provides great convenience in mathematics, statistics and engineering research areas. In the range of $(0, \pi / 2)$, the integrals given in terms of sine and cosine can easily be calculated with the help of Beta function. There are functional relationships between moments of Beta distribution. So, algebraic moments could be written. Hypergeometric function can be written with the help of Beta function and the Pochhammer symbol. Also, an Incomplete Beta function can be expressed as the form of hypergeometric functions by using Equation (18). X and Y are independent Gamma random variables with the parameters (α, λ) and (β, λ), respectively, the joint density of $U = X + Y$ and $V = \frac{X}{X + Y}$ could be computed. This is called the Beta density function. Distribution of the range of a random sample is a Beta variable with parameters n-1 and 2. In Table 1, incomplete Beta distribution is expressed how to show a relationship between Beta, Binomial, F, Geometric, Negative Binomial and Student's t-distributions.

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