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RESEARCH ARTICLE

ON SPECIAL LINEAR POLYNOMIAL SEQUENCES

Gopalan, M.A., Vidhyalakshmi, S. and *Thiruniraiselvi, N.

Department of Mathematics, SIGC, Trichy-620002, Tamilnadu, India

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ABSTRACT

This paper concerns with the study of obtaining an infinite sequence of linear polynomials such that the product of any two or three consecutive polynomials plus or minus their sum and increased by a polynomial of degree two with integer coefficients is a square of polynomial.

INTRODUCTION

A Set of positive integers ($a_1, a_2, a_3, ...a_m$) is said to have the property D(n), $n \in z - \{0\}$, if $a_i a_j + n$ is a perfect square for all $1 \le i \lhd j \le m$ and such a set is called a Diophantine –m-tuple with property D(n). Many mathematicians considered the problem of the existence of Diophantine quadruples with the property D(n) for any arbitrary integer n (Bashmakova, 1974) and also for any linear polynomial in n. Further, various authors considered the connections of the problems of diophantus, davenport and Fibonacci numbers in (Thamotherampillai, 1980; Brown, 1985; Gupta and Singh, 1985; Beardon and Deshpande, 2002; Deshpande, 2002; Deshpande, 2003; Bugeaud *et al.,* 2007; Tao Liqun, 2007; Fujita, 2008; Srividhya, 2009; Gopalan and Pandichelvi, 2011; Yasutsugu Fujita and Alain Togbe, 2011; Gopalan, 2012; Gopalan, 2012; Gopalan, 2012; Flipin *et al.,* 2012; Fujita, 2006; Filipin *et al.,* 2012; *Gopalan et al.,* 2014).

In this communication, we find special sequence of polynomials $S = \{a_0, a_1, a_2, ...\}$ in which the product of any two or three consecutive polynomials plus or minus their sum and increased by a polynomial of degree two with integer coefficients is a square of polynomial.

MATERIALS AND METHODS

Construction of Special Polynomial Sequence I

Let $a_0 = 2a+1$ and $a_1 = 4a+3$ be two linear polynomials such that $a_0 a_1 + a_0 + a_1 + (a^2 + 2a + 2) = (3a+3)^2$, a perfect square

Therefore (a_0, a_1) is the special dio-2-tuple with property $D(a^2 + 2a + 2)$

Let a_2 be a linear polynomial such that

$$(a_0 + 1)a_2 + (a^2 + 4a + 3) = p^2 \qquad \text{…………….} \quad (1)$$

***Corresponding author: Thiruniraiselvi, N.,**
Department of Mathematics, SIGC, Trichy-620002, Tamilnadu, India.

$$(a_1+1)a_2+(a^2+6a+5)=q^2 \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

Eliminating a_2 between (1) and (2), we get

$$(a_1+1)p^2-(a_0+1)q^2=(a^2+2a+1)(a_1-a_0) \quad \ldots\ldots\ldots\ldots\ldots.. \quad (3)$$

Introduction of the linear transformations

$$p=X+(a_0+1)T \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

$$q=X+(a_1+1)T \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

in (3) leads to the Pell equation

$$X^2=(a_0+1)(a_1+1)T^2+(a^2+2a+1) \quad \ldots\ldots\ldots\ldots\ldots\ldots. \quad (6)$$

Whose initial solution is $T_0=1, X_0=3a+3$

Thus (4) yields $p=5a+5$ and using (1), we get $a_2=12a+11$

Hence $(a_o,a_1,a_2)=(2a+1,4a+3,12a+11)$ is the required special dio-triple with property $D(a^2+2a+2)$

The repetition of the above process leads to the generation of special dio-3-tuples, namely,

$$(a_o,a_1,a_2),(a_1,a_2,a_3),\ldots\ldots$$

Note that the above results may be presented as a theorem as follows:

Theorem

Consider the infinite sequence $S=\{a_0,a_1,a_2,\ldots.\}$ of polynomials given by

$$a_{n+1}=a_n+2\omega_n^2(a+1) \quad \text{where} \quad \begin{array}{l}\omega_n=\omega_{n-2}+\omega_{n-1}, n=0,1,2,\ldots.\\ \omega_{-2}=0,\omega_{-1}=1\end{array}$$

This sequence is such that the product of any two or three consecutive polynomials added with their sum and increased by a polynomial of degree two with integer coefficients (a^2+2a+2) is a square of polynomial.

Replacing 'a' by a Gaussian integer and irrational number respectively in each of the above triples, it is noted that each resulting triple is a special Gaussian dio-3-tuple and irrational triple satisfying the required property.

A few examples are given below:

a	Dio-Triples			Property
	(a_o,a_1,a_2)	(a_1,a_2,a_3)	$a_2,a_3,a_4)$	
$1+i\sqrt{5}$	$\{(3+i2\sqrt{5}),$ $(7+i4\sqrt{5}),$ $(23+i2\sqrt{5})\}$	$\{(7+i4\sqrt{5}),$ $(23+i2\sqrt{5}),$ $(59+i30\sqrt{50}\}$	$\{(23+i2\sqrt{5}),$ $(59+i30\sqrt{5}),$ $(159+80i\sqrt{5})\}$	$D(4i\sqrt{5})$
$3+\mathrm{i}2$	$\{(7+\mathrm{i}4),$ $(15+\mathrm{i}8),$ $(47+\mathrm{i}24)\}$	$\{(15+\mathrm{i}8),$ $(47+\mathrm{i}24),$ $(119+i60)\}$	$\{(47+\mathrm{i}24),$ $(119+i60),$ $(319+i160)\}$	$D(13+i16$

Construction of Special Polynomial Sequence II

Let $a_0=2a+1$ and $a_1=5a+3$ be two linear polynomials such that $a_0a_1-(a_0+a_1)+(-a^2+2a+2)=(3a+1)^2$, a perfect square

Therefore (a_0, a_1) is the special dio-2-tuple with property $D(-a^2+2a+2)$
Let a_2 be a linear polynomial such that

$$(a_0-1)a_2-a^2+1=\alpha^2 \qquad \ldots\ldots\ldots\ldots.. \qquad (7)$$

$$(a_1-1)a_2-a^2-3a-1=\beta^2 \qquad \ldots\ldots\ldots\ldots.. \qquad (8)$$

Eliminating a_2 between (7) and (8), we get

$$(a_1-1)\alpha^2-(a_0-1)\beta^2=(-a^2+2a+1)(a_1-a_0) \qquad \ldots\ldots\ldots\ldots. \qquad (9)$$

Introduction of the linear transformations

$$\alpha = X+(a_0-1)T \qquad \ldots\ldots\ldots\ldots \qquad (10)$$

$$\beta = X+(a_1-1)T \qquad \ldots\ldots\ldots\ldots. \qquad (11)$$

in (9) leads to the Pell equation

$$X^2=(a_0-1)(a_1-1)T^2+(-a^2+2a+1) \qquad \ldots\ldots\ldots\ldots \qquad (12)$$

Whose initial solution is $T_0=1, X_0=3a+1$

Thus (10) yields $\alpha=5a+1$ and using (7), we get $a_2=13a+5$

Hence $(a_o, a_1, a_2)=(2a+1, 5a+3, 13a+5)$ is the required special dio-triple with property $D(-a^2+2a+2)$
The repetition of the above process leads to the generation of special dio-3-tuples, namely,

$$(a_o, a_1, a_2)=(2a+1, 5a+3, 13a+5),$$
$$(a_1, a_2, a_3)=(5a+3, 13a+5, 34a+13),$$
$$(a_2, a_3, a_4)=(13a+5, 34a+13, 89a+31),$$
$$(a_3, a_4, a_5)=(34a+13, 89a+31, 233a+81), \ldots.$$

Note that the above results may be presented as a theorem as follows:

Theorem

Consider the infinite sequence $S=\{a_0, a_1, a_2, \ldots.\}$ of polynomials given by

$$a_{n+1}=a_n+(\omega_n^2-\sigma_n^2)a+2\sigma_n^2 \quad \text{where} \quad \begin{array}{l} \omega_n=\omega_{n-2}+\omega_{n-1}, \\ \sigma_n=\sigma_{n-2}+\sigma_{n-1}, n=0,1,2,\ldots. \\ \omega_{-2}=1, \omega_{-1}=1, \sigma_{-2}=1, \sigma_{-1}=0 \end{array}$$

This sequence is such that the product of any two or three consecutive polynomials added with their sum and increased by a polynomial of degree two with integer coefficients $(-a^2+2a+2)$ is a square of polynomial

Conclusion

In this paper, we have presented a sequence of linear polynomials such that any set of 3 consecutive polynomials represents a special dio-triple with suitable property. To conclude, one may search for sequence of polynomials representing polygonal numbers and other special numbers leading to special dio-triples and quadruples with suitable property.

A few examples of special dio-triples are exhibited below

a	Dio-Triples			Property
	(a_o, a_1, a_2)	(a_1, a_2, a_3)	$a_2, a_3, a_4)$	
$1+i\sqrt{7}$	$\{(3+i2\sqrt{7}),$ $(8+i5\sqrt{7}),$ $(18+i13\sqrt{7})\}$	$\{(8+i5\sqrt{7}),$ $(18+i13\sqrt{7}),$ $(47+i34\sqrt{7}\}$	$\{(18+i13\sqrt{7}),$ $(47+i34\sqrt{7}),$ $(120+89i\sqrt{7})\}$	$D(10)$
$2+i\sqrt{7}$	$\{(5+i2\sqrt{7}),$ $(13+i5\sqrt{7}),$ $(31+i13\sqrt{7})\}$	$\{(13+i5\sqrt{7}),$ $(31+i13\sqrt{7}),$ $(81+i34\sqrt{7}\}$	$\{(31+i13\sqrt{7}),$ $(81+i34\sqrt{7}),$ $(209+89i\sqrt{7})\}$	$D(9-i2\sqrt{7})$
$1+\mathrm{i}4$	$\{(3+\mathrm{i}8),$ $(8+\mathrm{i}20),$ $(18+\mathrm{i}52)\}$	$\{(8+\mathrm{i}20),$ $(18+\mathrm{i}52),$ $(47+i136)\}$	$\{(18+\mathrm{i}52),$ $(47+i136),$ $(120+i356)\}$	$D(19)$
$6+\mathrm{i}4$	$\{(13+\mathrm{i}8),$ $(33+\mathrm{i}20),$ $(83+\mathrm{i}52)\}$	$\{33+\mathrm{i}20),$ $(83+\mathrm{i}52),$ $(217+i136)\}$	$\{(83+\mathrm{i}52),$ $(217+\mathrm{i}136),$ $(565+i356)\}$	$D(-6-i40)$

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