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RESEARCH ARTICLE

ADDITIVELY INVERSIVE SEMIRINGS/ HEMIRINGS CHARACTERIZED BY THEIR FUZZY IDEALS

***Nazish Kanwal and Muhammad Zakaullah Khan**

Department of Mathematics, Sir Syed University of Engineering and Technology, Block 5, University Road, Gulshan -e-Iqbal, Karachi-75300, Pakistan

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ABSTRACT

The main objective of this study is to construct fuzzy analogous using Zadeh's notion of the following results of Yusuf (Yusuf, 1965) regarding inversive semirings and hemirings. In case of a collection of left/right/two-sided ideals of an additively inversive semiring their non-empty intersection is also left/right/two-sided ideal. For any additively inversive hemiring, the complex sum of two left/right/two sided ideals is also a left/right/two-sided ideal. For an additively inversive hemiring S with zero, the complex sum of two left/right/two-sided ideals is the intersection of all left/right/two-sided ideals that contain the two ideals. For an additively inversive hemiring and its two left/right/two-sided ideals with identical intersections with the set of idempotent elements, the complex sum of the two ideals is the intersection of all left/right/two-sided ideals that contain the two ideals. The following theorems manifest the fuzzy analogous of the above results.

(i') For any collection of left/right/two-sided fuzzy ideals of an additively inversive semiring, their non-empty intersection is also a left/right/two-sided fuzzy ideal.

(ii') For an additively inversive hemiring, the complex sum of two left/right/two-sided fuzzy ideals is also a left/right/two-sided fuzzy ideal.

(iii') For an additively inversive hemiring containing absorbing zero '0' and two fuzzy ideals satisfying 0-1 condition, the complex sum of two left/right/two sided fuzzy ideals is the intersection of all left/right/two sided fuzzy ideals that contain the two ideals.

(iv') For an additively inversive hemiring containing absorbing zero '0' and two fuzzy ideals with identical intersections with the set of idempotent elements, satisfying 0-1 condition, the complex sum of two left/right/two sided fuzzy ideals is the intersection of all left/right/two-sided fuzzy ideals that contain the two ideals.

It is worth mentioning that the fundamental concept of a fuzzy set, introduced by Zadeh in 1965, has been applied by many authors to generalize some of the basic notions of algebra.

INTRODUCTION

In 1965, Lofti A. Zadeh (1965) introduced the notion of a fuzzy subset of a crisp set as a method for representing uncertainty. Fuzzy set theory was mathematically formulated by the assumption that classical sets were not appropriate or natural in describing the real life problems. Fuzzy set theory has greater richness in applications than the ordinary set theory. Thus far attention has been drawn to generalize the basic concepts of classical algebra to fuzzy sets resulting in the development of the theory of fuzzy algebra. The study starts with the fundamental concepts of a fuzzy set based on Zadeh's classical paper Lofti A. Zadeh (1965) which he introduced in 1965. This paper provides a natural framework for generalizing some of the basic notions of algebra. The element of the theory of fuzzy groups was formulated by Kuroki (1979); Kuroki (1981). Liu (1982) and Liu (1983) investigated fuzzy subrings and fuzzy ideals of a ring. The study of fuzzy modules was initiated by Pan (1987) and Golan (1989). This study develops some results in the fuzzy settings about additively inversive semirings / hemirings which were proved for ordinary additively inversive semirings / hemirings by Yusuf (1965). These results pertain to the intersection and sum of fuzzy ideals. Section 1 introduces the basic terminology to be used in the next sections. Section 2 discusses results about the arbitrary intersection and the complex sum of the fuzzy ideals of additively inversive semirings / hemirings. These are the generalizations of some of the results of Yusuf (1965) in the ordinary settings. Section 3 proves several results about the relationship between the complex sum and the intersection of fuzzy ideals of additively inversive semirings / hemirings. Finally, section 4 concludes the study.

****Corresponding author: Nazish Kanwal,*** *Department of Mathematics, Sir Syed University of Engineering and Technology, Block 5, University Road, Gulshan -e-Iqbal, Karachi-75300, Pakistan*

§1 Preliminaries

Throughout this paper, S will denote a semiring and $E^{+}$the setof all idempotents of the additively inversive hemirings.

Definition

An additively inversive semiring is one in which the additive semigroup is an inversive semigroup, that is, for each $a \in S$, there exists a unique element $a' \in S$ such that $a + a' + a = a$ and $a' + a + a' = a'$.

Example:

Let $S = \{0, a, b\}$. The following tables define the two binary operations on an additively Inversive semiring S.

Table 1

+	o	a	b
o	o	a	b
a	a	o	b
b	b	b	b

Table 2

o	o	a	b
o	o	o	o
a	o	o	o
b	o	o	b

Remark:

LaTorre (1962) proved that for an additively inversive semiring S and $a, b \in S$

$$(ab)' = a'b = ab', \quad ab = a'b',$$

where $a', b' \in S$ are the additive inverses of a and b, respectively.

Definition

A non-empty set A of an *additively inversive semiring* S is called a *left (right) ideal* if and only if

(i) $a, b \in A \Rightarrow a + b \in A$,
(ii) $a \in A \Rightarrow a' \in A$,
(iii) $a \in A$, $s \in S \Rightarrow sa(as) \in A$.

Definition

A non-empty set A of an *additively inversive semiring* S is called a *two-sided ideal* if it is both a right as well as a left ideal of S.

Definition

A subset λ of an additively inversive semiring / hemiring is called a *fuzzy ideal of an additively inversive semiring / hemiring*, if the following conditions hold for each a, b of an additively inversive semiring / hemiring:

(i) $\lambda(a - b) \geq min\{\lambda(a), \lambda(b)\}$, where $(a - b)$ is an element of an additively inversive semiring / hemiring such that

$a + b + a = a$ and $b + a + b = b$.

(ii) $\lambda(ab) \geq max\{\lambda(a), \lambda(b)\}$.

Definition

The identity element 'e' is called the *absorbing zero* of an additively inversive hemiring, if

$a + e = e + a = a$ and $a \cdot e = e \cdot a = e$ for all $a \in S$.

Definition

The sets of idempotent elements of the additively inversive hemiring S, for each $x \in S$, are defined as

$E^{[+]} = \{x \in S : x + x = x\}$

and

$E^{[\cdot]} = \{e \in S : e \cdot e = e\}$.

Also, we define

$V^{[+]} = \{ a \in S : x + a + x = x$ and $a + x + a = a \}$

and

$V^{[\cdot]} = \{ b \in S : x \cdot b \cdot x = x$ and $b \cdot x \cdot b = b \}$.

Proposition

If S is a semiring with an absorbing zero and λ , μ are fuzzy ideals of S, then λ+μ is the smallest fuzzy ideal of Scontaining both λ and μ , (Remark 2.2, Ahsan et al., 2011).

Definition

Let S be an additively inversive hemiring Containing an absorbing zero '0' and λ, μ are fuzzy ideals of S satisfying $\lambda(0) = 1$ and $\mu(0) = 1$, is the 0-1 condition.

§2 Arbitrary Intersection and Complex Sum of Fuzzy Ideals

Here in this section we give the results of the arbitrary intersection and the complex sum of fuzzy ideals of additively inversive semirings / hemirings in ordinary setting, as described and proved in Yusuf (1965).

Table 3

+	a	b	c
o	a	a	a
a	a	b	b
b	a	b	c

Table 4

o	a	b	c
a	b	b	b
b	b	b	b
c	b	b	b

Theorem

Let $\{A_i\}$ denote any collection of left [two-sided] ideals of an additively inversive semiring S. If the intersection of A_i is non-empty, then it is a left [two-sided] ideal of S.

Theorem

Let S be an additively inversive hemiring. Then the complex sum $A_1 + A_2$ of two left [two-sided] ideals A_1 and A_2 is a left [two-sided] ideal.

Remark

We note that the complex sum $A_1 + A_2$ of left ideals is contained in every left ideal that contains both A_1 and A_2 but $A_1 + A_2$ may not itself contain A_1 and A_2 as is clear from the following tables:

Now the sets $\{a, b\}$ and $\{b, c\}$ are ideals of S. The complex sum is $\{a, b\} + \{b, c\} = \{a, b\}$ and it does not contain the ideal $\{b, c\}$. The following theorems manifest the above results in fuzzy settings.

Theorem

Let $\{\lambda_i\}$ denote any collection of left [two-sided] fuzzy ideals of an additively inversive semiring S. If the intersection of λ_i is non-empty, then it is a left [two-sided] fuzzy ideal of S.

Proof

We have

$$\left(\bigcap_{i\in I} \lambda_i\right)(x) = inf_{i\in I}\{\lambda_i(x)\}\forall x\epsilon X.$$

For any $x, y\epsilon S$,

$$\left(\bigcap_{i\in I} \lambda_i\right)(x-y) = inf_{i\in I}\{\lambda_i(x-y)\}$$

$$\geq inf_{i\in I}\{\lambda_i(x), \lambda_i(y)\}$$
$$\geq \min[inf_{i\in I}\{\lambda_i(x)\}, inf_{i\in I}\{\lambda_i(y)\}]$$
$$\geq \min\left\{\left(\bigcap_{i\in I} \lambda_i\right)(x), \left(\bigcap_{i\in I} \lambda_i\right)(y)\right\}.$$

Again,

$$\left(\bigcap_{i\in I} \lambda_i\right)(xy) = inf_{i\in I}\{\lambda_i(xy)\}$$

$$\geq \max\ [inf_{i\in I}\{\lambda_i(x)\}, inf_{i\in I}\{\lambda_i(y)\}]$$

$$\geq \max\left\{\left(\bigcap_{i\in I} \lambda_i\right)(x), \left(\bigcap_{i\in I} \lambda_i\right)(y)\right\}.$$

Hence, intersection of $\{\lambda_i\}$ is a fuzzy ideal of S. □

Theorem

Let S be an additively inversive hemiring. Then the complex sum $\lambda + \mu$ of two left [two-sided] fuzzy ideals λ and μ is a left [two-sided] fuzzy ideal.

Proof

For any $x, y \epsilon S$,

we have

$$(\lambda + \mu)(x) = \sup\quad [min\{\lambda(a), \mu(b)\} : x = a + b]$$

and

$(\lambda + \mu)(y) = \sup \quad [min\{\lambda(c), \mu(d)\} : y = c + d]$.

$\min \quad \{(\lambda + \mu)(x), (\lambda + \mu)(y)\}$

$$= min[sup\{min\{\lambda(a), \mu(b)\}\}, sup\{min\{\lambda(c), \mu(d)\}\}]$$

$$= sup[min\{min\{\lambda(a), \mu(b)\}\}, min\{min\{\lambda(c), \mu(d)\}\}],$$

(using the infinite meet distributive law).

$$= sup[min\{min\{\lambda(a), \lambda(c)\}\}, min\{min\{\mu(b), \mu(d)\}\}]$$

$\leq sup[\min\{ min\{\lambda(a-c)\}, min\{\mu(b-d)\}\}]$,

(by Abou-Draeb, (2000)).

$\leq sup[min\{\lambda(a-c), \mu(b-d)\}]$

$\leq (\lambda + \mu)(z), \qquad z = (a+b) - (c+d) = x - y$

$\leq (\lambda + \mu)(x - y)$.

Also,

$(\lambda + \mu)(x) = sup[min\{\lambda(a), \mu(b)\} : x = a + b]$

$\leq sup[min\{\lambda(ay), \mu(by)\}]$
$\leq sup[min\{\lambda(u), \mu(v)\} : xy = u + v]$
$\leq (\lambda + \mu)(xy). \qquad \ldots \; (I)$

Again,

$(\lambda + \mu)(y) = sup[min\{\lambda(c), \mu(d)\} : y = c + d]$

$\leq sup[min\{\lambda(xc), \mu(xd)\}]$
$\leq sup[min\{\lambda(w), \mu(z)\} : xy = w + z]$
$\leq (\lambda + \mu)(xy). \qquad \ldots \; (II)$

From (I) and (II), we have

$(\lambda + \mu)(xy) \geq \max\{(\lambda + \mu)(x), (\lambda + \mu)(y)\}$.

Thus, $\lambda + \mu$ is an ideal of an additively inversive hemiring S. □

§3 Relationship between the Intersection and Complex Sum of Fuzzy Ideals

In this section we fuzzify the results, proved in (Yusuf, 1965) about the relationship between the complex sum and the intersection of fuzzy ideals of additively inversive semirings / hemirings.

Theorem

If an additively inversive hemiring S contains a zero element z, then the complex sum $A_1 + A_2$ of the left [two-sided] ideals A_1 and A_2 is the intersection of all left [two-sided] ideals that contain A_1 and A_2.

Theorem

Let S be an additively inversive hemiring. Let A_1 and A_2 be left [two-sided] ideals of S. Suppose that $A_1 \cap E^+ = A_2 \cap E^+$. Then the complex sum $A_1 + A_2$ is the intersection of all left [two-sided] ideals that contain A_1 and A_2.
The analogous results for right ideals also hold.
The following theorems manifest the above results in fuzzy settings.

Theorem

If an additively inversive hemiring S contains an absorbing zero '0' and λ, μ are fuzzy ideals of S satisfying $\lambda(0) = 1$ and $\mu(0) = 1$, then the complex sum $\lambda + \mu$of the left [two-sided] fuzzy ideals λ and μ is the intersection of all left [two-sided] fuzzy ideals that contain $\lambda$and μ.

Proof: Since S contains an absorbing zero, for each $a \in S, a + 0 = a = 0 + a$, So,

$\lambda(a + 0) \geq min\{\lambda(a), \lambda(0)\} = \lambda(a),$

and

$$\mu(a + 0) \geq min\{\mu(a), \mu(0)\} = \mu(a).$$

Also,

$\lambda(a \cdot 0) \geq max\{\lambda(a), \lambda(0)\} = \lambda(0) = 1,$

and

$$\mu(a \cdot 0) \geq max\{\mu(a), \mu(0)\} = \mu(0) = 1.$$

We also note that,

$(\lambda + \mu)(a) = \sup\ [\min\{\lambda(a), \mu(0)\} : a = a + 0\ ] \geq \lambda(a).$

Thus, $\lambda + \mu \geq \lambda$.

Similarly,

$(\lambda + \mu)(a) = \sup\ [\min\{\lambda(0), \mu(a)\} : a = 0 + a\ ] \geq \mu(a).$

Thus,

$\lambda + \mu \geq \mu.$

If ξ is a left [two-sided] fuzzy ideal of S containing both λ and μ, then $\lambda + \mu \leq \xi$.

From from second Theorem in §2, we have that $\lambda + \mu$ is a left [two-sided] fuzzy ideal and, since $\lambda + \mu$ is contained in every left [two-sided] fuzzy ideal that contains λ and μ, It follows that $\lambda + \mu$ is the intersection of all left [two-sided] fuzzy ideals that contain λ and μ.

Corollary

Let λ be the fuzzy ideal of an additively inversive hemiring S and E^+ be the set of all idempotents of S, then $\lambda \cap E^+$ is the fuzzy ideal of S.

Proof

By Golan (1992), E^+ is the ideal of S.

To prove that $\lambda \cap E^+$ is a fuzzy ideal of S, we must show that, E^+ is the fuzzy ideal of S.
Let δ be the fuzzy subset of E^+, we need to show that
$\delta(x + y) \geq \min\{\delta(x), \delta(y)\}$ and $\delta(xy) \geq \max\{\delta(x), \delta(y)\}$.
Let $x, y \in E^+$, therefore $x + x = x$ and $y + y = y$.
Since E^+ is an ideal of an additively inversive hemiring S, therefore,
$x + y + x = x$ and $y + x + y = y$, for each $x, y \in E^+$.

Also, $x + y = (x + y) + (x + y)$.

Now

$$x + y + x = (x + x + y) = (x + y).$$

$x + y \geq x$ and $x + y \geq y$

$\Rightarrow \delta(x + y) \geq \delta(x)$ and $\delta(x + y) \geq \delta(y)$

Or, $\delta(x) \leq \delta(x + y)$ and $\delta(y) \leq \delta(x + y)$

$\Rightarrow \min\{\delta(x), \delta(y)\} \leq \delta(x + y)$
$\Rightarrow \delta(x + y) \geq \min\{\delta(x), \delta(y)\}$. ... (III)

Also,

$xy = xy + xy = x(y + y)$
$\Rightarrow xy \geq x$ and $xy \geq y$
$\Rightarrow \delta(xy) \geq \delta(x)$ and $\delta(xy) \geq \delta(y)$
$\Rightarrow \delta(xy) \geq \max\{\delta(x), \delta(y)\}$. ... (IV)

Hence, (III) and (IV) show that, δ is the fuzzy ideal of E^+.
The intersection of two fuzzy ideals is again an ideal, therefore, $\lambda \cap E^+$ is the fuzzy ideal of S. □

Theorem

Let S be an additively inversive hemiring having an absorbing zero. Let λ and μ be left [two-sided] fuzzy ideals of S satisfying $\lambda(0) = 1$ and $\mu(0) = 1$. Suppose that $\lambda \cap E^+ = \mu \cap E^+$. Then the complex sum $\lambda + \mu$ is the intersection of all left [two-sided] fuzzy ideals that contain λ and μ.

Proof

Since for each $x \in S$, there exists $a \in S$, such that
$x = x + a + x$.

We also note that

$$\lambda(x + a + x) \geq min\{\lambda(x), \lambda(a + x)\} = \lambda(x)$$

and

$$\mu(x + a + x) \geq min\{\mu(x), \mu(a + x)\} = \mu(x).$$

Therefore, by Proposition of §1,

$$\lambda + \mu \geq \lambda \quad \text{and} \quad \lambda + \mu \geq \mu.$$

If ξ is a left [two-sided] fuzzy ideal of S containing both λ and μ, then $\lambda + \mu \leq \xi$.
from second Theorem in §2, we have that $\lambda + \mu$ is a left [two-sided] fuzzy ideal and, since $\lambda + \mu$ is contained in every left [two-sided] fuzzy ideal that contains λ and μ, it follows that $\lambda + \mu$ is the intersection of all left [two-sided] fuzzy ideals that contain λ and μ. □

§4 Conclusion

This paper gives the techniques of fuzzification of the theorems on ideals in ordinary settings which paves the way for fuzzifying other algebraic structures.

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